

ASSEMBLY MAPS FOR GROUP EXTENSIONS IN K -THEORY AND L -THEORY WITH TWISTED COEFFICIENTS

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ABSTRACT. In this paper we show that the Farrell-Jones isomorphism conjectures are inherited in group extensions for assembly maps in K -theory and L -theory with twisted coefficients.

INTRODUCTION

Under what assumptions are the Farrell-Jones isomorphism conjectures inherited by group extensions or subgroups? We will formulate a version of the standard conjectures (see Farrell-Jones [10]) with twisted coefficients in an additive category, and then study these questions via the continuously controlled assembly maps of [11, §7]. A formulation using the Davis-Lück assembly maps [9] has already been given by Bartels and Reich [4], and applied there to show inheritance by subgroups. Recall that the Farrell-Jones conjecture in algebraic K -theory asserts that certain “assembly” maps

$$H_n^G(E_{\mathcal{VC}}G; \mathbb{K}_R) \rightarrow K_n(RG)$$

are isomorphisms, for a given ring R , and all $n \in \mathbf{Z}$. Here the space $E_{\mathcal{VC}}G$ is the universal G -CW-complex for G -actions with virtually cyclic isotropy, and the left-hand side denotes equivariant homology with coefficients in the non-connective K -theory spectrum for the ring R .

Theorem A. *Let $N \rightarrow G \xrightarrow{\pi} K$ be a group extension, where $N \triangleleft G$ is a normal subgroup, and K is the quotient group. Let $\mathcal{A}$ be an additive category with G -action. Suppose that*

- (i) *The group K satisfies the Farrell-Jones conjecture in algebraic K -theory, with twisted coefficients in any additive category with K -action.*
- (ii) *Every subgroup of G containing N as a subgroup, with virtually cyclic quotient, satisfies the Farrell-Jones conjecture in algebraic K -theory, with twisted coefficients in $\mathcal{A}$.*

Then the group G satisfies the Farrell-Jones conjecture in algebraic K -theory, with twisted coefficients in $\mathcal{A}$.

This is a special case of a more general result (see Theorem 4.7). The same statement holds for algebraic L -theory as well, where the coefficient categories are additive categories with involution. The corresponding result for the Baum-Connes conjecture was

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obtained by Oyono-Oyono [12], and our proof follows the outline given there. One of the main points is that the most effective methods known for proving the standard Farrell-Jones conjectures (for particular groups G) also work for the twisted coefficient versions (compare [1], [3], [6], [7], [15], [16], and [17]). An immediate corollary to Theorem A is the following.

Corollary (Corollary 4.10). *The Farrell-Jones conjecture with twisted coefficients is true for $G_1 \times G_2$ if and only if it is true for G_1 , G_2 , and every product $V_1 \times V_2$, where $V_1 \leq G_1$ and $V_2 \leq G_2$ are virtually cyclic subgroups.*

The fibered isomorphism conjecture of Farrell and Jones [10] for a group G and a ring R asserts that for every group homomorphism, $\phi: H \rightarrow G$, the assembly map for H relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, is an isomorphism. This conjecture implies the Farrell-Jones conjecture and has better inheritance properties. For example, the fibered version of our Theorem A is also true (see, for example, [2, Section 2.3]). The following result shows that the Farrell-Jones conjecture with twisted coefficients implies the Fibered Farrell-Jones conjecture.

Theorem B. *Suppose that $\phi: H \rightarrow G$ is a group homomorphism. Then the Farrell-Jones conjecture for G , with twisted coefficients in any G -category, implies that the assembly map for H relative to the family generated by the subgroups $\phi^{-1}(V)$, $V \subset G$ virtually cyclic, is an isomorphism with twisted coefficients in any H -category.*

The corresponding result for the Davis-Lück assembly maps was obtained by Bartels-Reich [4], who also pointed out a number of applications of the assembly map with twisted coefficients, including the study the K - and L -theory of twisted group rings (see also Example 4.8 and Example 4.9 below). One can check as in [11] that those assembly maps are equivalent to the continuously controlled assembly maps used in this paper.

1. ASSEMBLY VIA CONTROLLED CATEGORIES

The controlled categories of Pedersen [13], Carlsson-Pedersen [6], [8] are our main tool for identifying various different assembly maps. We will recall the definition of these categories, and then the usual assembly maps are obtained by applying functors

$$H: G\text{-CW-Complexes} \rightarrow \text{Spectra}$$

as described in [11]. We will extend the earlier definitions in order to allow an additive category as coefficients, instead of just working with modules over a ring R . A formulation for assembly maps with coefficients in the setting of [9] has already been given in [4]. Following the method of [11], one can check that the two different descriptions give the same assembly maps.

Let G be any discrete group, and let X be a G -CW complex (we will use a left G -action). Subspaces of the form $G \cdot D \subset X$, with D compact in X , are called G -compact subspaces of X . More generally, a subspace whose closure has this form is called relatively G -compact. A *resolution* of X is a pair $(\bar{X}, p)$, where $\bar{X}$ is a free G -CW complex and $p: \bar{X} \rightarrow X$ is a continuous G -equivariant map, such that for every G -compact set $G \cdot D \subset X$ there exists

a G -compact set $G \cdot \overline{D} \subset \overline{X}$ such that $p(G \cdot \overline{D}) = G \cdot D$. The notion of resolution comes from [13], and was developed further in [1, §3]. The original example was $\overline{X} = G \times X$, with the diagonal G -action and first factor projection.

Let $\mathcal{A}$ be an additive category with involution, and suppose that $\mathcal{A}$ has a right G -action compatible with the involution. This is a collection of covariant functors $\{g^*: \mathcal{A} \rightarrow \mathcal{A}, \forall g \in G\}$, such that $(g \circ h)^* = h^* \circ g^*$ and $e^* = id$. We require that the functors g^* commute with the involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ (an involution is a contravariant functor with square the identity).

Definition 1.1. Let (Z, X) be a G -CW pair, where X is a closed G -invariant subspace. Let $Y = Z - X$, and fix a resolution $p: \overline{Z} \rightarrow Z$, whose restriction to Y is denoted $\overline{Y}$. The category $\mathcal{D}(Z, X; \mathcal{A})$ has objects $A = (A_y)$ consisting of a collection of objects of $\mathcal{A}$, indexed by $y \in \overline{Y}$, and morphisms $\phi: A \rightarrow B$ consisting of collections $\phi = (\phi_y^z)$ of morphisms $\phi_y^z: A_y \rightarrow B_z$ in $\mathcal{A}$, indexed by $y, z \in \overline{Y}$, satisfying:

- (i) the support $\{y \in \overline{Y} \mid A_y \neq 0\}$ is *locally finite* in $\overline{Y}$, and relatively G -compact in $\overline{Z}$.
- (ii) for each morphism $\phi: A \rightarrow B$, and for each $y \in \overline{Y}$, the set $\{z \mid \phi_y^z \neq 0 \text{ or } \phi_z^y \neq 0\}$ is finite.
- (iii) the morphisms $\phi: A \rightarrow B$ are *continuously controlled* at $X \subset Z$. For every $x \in X$, and for every G_x -invariant neighbourhood U of x in Z , there is a G_x -invariant neighbourhood V of x in Z so that $\phi_y^z = 0$ and $\phi_z^y = 0$ whenever $p(y) \in (Y - U)$ and $p(z) \in (V \cap U \cap Y)$.

If $X = \emptyset$, we use the shorter notation $\mathcal{D}(Z; \mathcal{A}) := \mathcal{D}(Z, \emptyset; \mathcal{A})$, and in this case the continuous control condition (iii) on morphisms is vacuous. If S is a discrete left G -set, we denote by $\mathcal{D}_l(S \times Z, S \times X; \mathcal{A})$ the subcategory where the morphisms are S -level-preserving: $\phi_{(s,y)}^{(s',z)} = 0$ if $s \neq s' \in S$, for any $y, z \in Y$.

The category $\mathcal{D}(Z, X; \mathcal{A})$ is an additive category with involution, where the dual of A is given by $(A^*)_y = A_y^*$ for all $y \in \overline{Y}$. It depends functorially on the pair (Z, X) of G -CW complexes. The actions of G on $\mathcal{A}$ and Z induce a right G -action on $\mathcal{D}(Z, X; \mathcal{A})$. For $g \in G$, we set $(gA)_y = g^* A_{gy}$ and $(g\phi)_y^z = g^*(\phi_{gy}^{gz})$. The fixed subcategory will be denoted $\mathcal{D}^G(Z, X; \mathcal{A})$. If $G = \{e\}$ is the trivial group, we use the notation $\mathcal{D}^0(Z, X; \mathcal{A})$. We have not included the resolution $(\overline{Z}, p)$ in the notation, because two different resolutions give G -equivalent categories (see [1, Prop. 3.5]). We can compare these fixed subcategories to the equivariant category $\mathcal{B}_G(Z, X; R)$ defined in [11, §7].

Lemma 1.2. *There is an equivalence of categories $\mathcal{B}_G(Z, X; R) \simeq \mathcal{D}^G(Z, X; \mathcal{A})$, when $\mathcal{A}$ is the category of finitely-generated free R -modules.*

Proof. We define a functor $F: \mathcal{D}^G(Z, X; \mathcal{A}) \rightarrow \mathcal{B}_G(Z, X; R)$ by sending an object A to the free R -module $F(A)_y = \bigoplus_{g \in G_y} A_{(g,y)}$, for all $y \in Y$, with the obvious reference map to Y . Similarly, for a morphism $\phi: A \rightarrow B$, we define $F(\phi)_y^z = (\phi_{g,y}^{g',z})_{g,g' \in G}$, for all $y, z \in Y$. The verification that this definition makes sense will be left to the reader.

Conversely, we can define a functor $F': \mathcal{B}_G(Z, X; R) \rightarrow \mathcal{D}^G(Z, X; \mathcal{A})$ on objects by decomposing an object $A = (A_y)$ of $\mathcal{B}_G(Z, X; R)$ as $A_y = \bigoplus_{g \in G_y} (A_y)_g$, since A_y is a

finitely-generated free RG_y -module. Now we let $F'(A)_{(g,y)} = (A_y)_g$, for all $y \in Y$, $g \in G$, and on morphisms by letting $F'(\phi)_{g,y}^{g',z} = \phi_{gy}^{g',z}$. Again the verifications will be left to the reader (technically we should work with a category equivalent to $\mathcal{B}_G(Z, X; R)$, in which the objects are based: each $A = R[T]$, where T is a free G -set, and T is equipped with a reference map to $X \times [0, 1]$). $\square$

For applications to assembly maps, we will let X be a G -CW complex and $Z = X \times [0, 1]$ so that $Y = X \times [0, 1)$. The category just defined will be denoted

$$\mathcal{D}^G(X \times [0, 1); \mathcal{A}) := \mathcal{D}^G(X \times [0, 1], X \times 1; \mathcal{A}) .$$

Let $\mathcal{D}^G(X \times [0, 1); \mathcal{A})_\emptyset$ denote the full subcategory of $\mathcal{D}^G(X \times [0, 1); \mathcal{A})$ with objects A such that the intersection with the closure

$$\text{supp}(A) = \overline{\{(x, t) \in \overline{X} \times [0, 1) \mid A_{(x,t)} \neq 0\}} \cap (X \times 1)$$

is the empty set.

Example 1.3. If $\mathcal{A}$ is the additive category of finitely generated free R -modules, then $\mathcal{D}^G(X \times [0, 1); \mathcal{A})_\emptyset$ is equivalent to the category of finitely generated free RG -modules, for any G -CW complex X .

The quotient category will be denoted $\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}$, and we remark that this is a germ category (see [11, §7], [14], [6]). The objects are the same as in $\mathcal{D}^G(X \times [0, 1); \mathcal{A})$ but morphisms are identified if they agree close to $\overline{X} = \overline{X} \times 1$ (i.e. on the complement of a neighbourhood of $\overline{X} \times 0$). Here is a useful remark.

Lemma 1.4 ([11]). *Let S be a discrete left G -set. The forgetful functor*

$$\mathcal{D}_l^G(S \times X \times [0, 1); \mathcal{A})^{>0} \rightarrow \mathcal{D}^G(S \times X \times [0, 1); \mathcal{A})^{>0}$$

is an equivalence of categories.

Proof. In the germ category, every morphism has a representative which is level-preserving with respect to projection on S . $\square$

The category $\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}$ is an additive category with involution, and we obtain a functor $G\text{-CW-Complexes} \rightarrow \text{AddCat}^-$. The results of [5, 1.28, 4.2] now show that the functors $F^\lambda: G\text{-CW-Complexes} \rightarrow \text{Spectra}$ defined by

$$(1.5) \quad F_G^\lambda(X; \mathcal{A}) := \begin{cases} \mathbb{K}^{-\infty}(\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}) \\ \mathbb{L}^{-\infty}(\mathcal{D}^G(X \times [0, 1); \mathcal{A})^{>0}) \end{cases} ,$$

where $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$ respectively, are G -homotopy invariant and G -excisive.

We can now extend the definition of the assembly maps to allow coefficients in any additive category with G -action.

Definition 1.6. We define the *continuously controlled assembly map with coefficients in $\mathcal{A}$* to be the map $F_G^\lambda(X; \mathcal{A}) \rightarrow F_G^\lambda(\bullet; \mathcal{A})$.

From the methods of [11], the continuously controlled assembly map with coefficients is homotopy equivalent to the assembly map with coefficients constructed in [4]. The most important example to consider is when $X = E_{\mathcal{VC}}G$, in which case the *Farrell-Jones conjecture with coefficients* asserts that this assembly map is an equivalence. Given a discrete group G , a family of subgroups $\mathcal{F}$ of G , and coefficients $\mathcal{A}$, we will refer to

$$F_G^\lambda(E_{\mathcal{F}}G; \mathcal{A}) \rightarrow F_G^\lambda(\bullet; \mathcal{A})$$

as the $(G, \mathcal{F}, \mathcal{A})$ -assembly map.

By applying $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$ to the sequence of additive categories (with involution):

$$\mathcal{D}^G(X \times [0, 1]; \mathcal{A})_\emptyset \rightarrow \mathcal{D}^G(X \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}^G(X \times [0, 1]; \mathcal{A})^{>0}$$

we obtain a fibration of spectra [6]. As in [11], we have the following description for the assembly map.

Theorem 1.7 ([11, §7]). *The continuously controlled assembly map*

$$F_G^\lambda(X; \mathcal{A}) \rightarrow F_G^\lambda(\bullet; \mathcal{A})$$

is homotopy equivalent to the connecting map

$$\lambda(\mathcal{D}^G(X \times [0, 1]; \mathcal{A})^{>0}) \rightarrow \Omega^{-1}\lambda(\mathcal{D}^G(X \times [0, 1]; \mathcal{A})_\emptyset)$$

for $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$.

See [11, §2] for the definition of homotopy equivalent functors from

$$G\text{-}CW\text{-Complexes} \rightarrow Spectra,$$

and [9, 5.1] for the result that any functor $E: \mathbf{Or}(G) \rightarrow Spectra$ out of the orbit category of G may be extended uniquely (up to homotopy) to a functor $E_\%: G\text{-}CW\text{-Complexes} \rightarrow Spectra$ which is G -homotopy invariant and G -excisive. This will be our method for comparing functors. The *orbit category* $\mathbf{Or}(G)$ is the category with objects G/K , for K any subgroup of G , and the morphisms are G -maps.

2. CHANGE OF COEFFICIENTS

We will need some ‘change of coefficient’ properties for the categories defined in the last section. The first three properties are essentially just translations of [4, Proposition 2.8] into our language. The corresponding versions for additive categories with involution are needed to apply these change of coefficient functors to L -theory.

Definition 2.1. Let K and G be groups, $\mathcal{A}$ an additive category with commuting right K and G -actions, and S a K - G biset. Then, the category $\mathcal{D}^K(S; \mathcal{A})$ has a right G -action via $(g \cdot A)_y = g^* A_{yg^{-1}}$ and $(g \cdot \phi)_y^z = g^* \phi_{yg^{-1}}^{zg^{-1}}$, for all $y, z \in \bar{S}$. We will mostly use the level-preserving subcategory $\mathcal{D}_l^K(S; \mathcal{A})$.

If T is a left G -set, and S is a transitive K - G biset (meaning that $K \backslash S / G$ is a point), we define a $K \times G$ -action on $S \times T$ by the formula $(k, g) \cdot (s, t) := (ks g^{-1}, gt)$ for all $(k, g) \in K \times G$ and all $(s, t) \in S \times T$. This action is used in the statements below.

Lemma 2.2. *Let T be a left G -set, and S be a transitive K - G biset. Then there is an additive functor*

$$F: \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))$$

which induces an equivalence of categories

$$\mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \simeq \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})) .$$

Proof. We will take the standard resolutions $\bar{S} = K \times S$, with elements denoted (k, s) , for $k \in K$ and $s \in S$, and $\bar{T} = G \times T \times [0, 1]$, with elements denoted (g, t) , for $g \in G$ and $t \in T \times [0, 1]$. Therefore

$$\bar{S} \times \bar{T} = K \times G \times S \times T \times [0, 1]$$

is a resolution for $S \times T \times [0, 1]$. We define the functor

$$F: \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))$$

on objects by setting $B = F(A)_{(g,t)}$ in $\mathcal{D}_l^K(S; \mathcal{A})$ as the object $B = (B_{(k,s)})$ with $B_{(k,s)} = A_{(k,g,s,t)}$ in $\mathcal{A}$. We use a similar formula for morphisms:

$$\left(F(\phi)_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')} = \phi_{(k,g,s,t)}^{(k',g',s',t')}$$

The proof that this is a well-defined functor is given in Section 5, where step (5''') of the argument depends on the assumption that S is a transitive K - G biset.

Since $\mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \simeq \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A})_\emptyset$ and $\mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})) \simeq \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))_\emptyset$, the functor F induces an additive functor

$$F: \mathcal{D}_l^{K \times G}(S \times T; \mathcal{A}) \rightarrow \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})).$$

On this subcategory, we define an inverse additive functor

$$F': \mathcal{D}_l^G(T; \mathcal{D}_l^K(S; \mathcal{A})) \rightarrow \mathcal{D}_l^{K \times G}(S \times T; \mathcal{A})$$

on objects by setting $F'(B)_{(k,g,s,t)} = (B_{(g,t)})_{(k,s)}$, and a similar formula for morphisms:

$$F'(\phi)_{(k,g,s,t)}^{(k',g',s',t')} = \left(\phi_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')}$$

It is easy to check that F' is a well-defined functor. The functors F and F' are inverses, so give an equivalence of categories. $\square$

Corollary 2.3. *Let G and K be groups, and $\mathcal{A}$ be an additive category with commuting right K and G -actions,. Then*

$$\mathcal{D}^{K \times G}(\bullet; \mathcal{A}) \simeq \mathcal{D}^G(\bullet; \mathcal{D}^K(\bullet; \mathcal{A})) .$$

Proof. We substitute $S = \bullet$ and $T = \bullet$ in the statement above. Note that morphisms are automatically level-preserving in this case. $\square$

Lemma 2.4. *Let K and G be groups, $\mathcal{A}$ an additive category with commuting right K and G -actions, and S a transitive K - G biset. Then, for any G -CW complex X , the functors*

$$F_{K \times G}^\lambda(S \times X; \mathcal{A})$$

and

$$F_G^\lambda(X; \mathcal{D}_l^K(S; \mathcal{A}))$$

are homotopy equivalent, where $\lambda = \mathbb{K}^\infty$ or $\mathbb{L}^\infty$. Here $K \times G$ acts on $S \times X$ by the formula $(k, g) \cdot (x, s) := (ksg^{-1}, gx)$.

Proof. By [9, 5.1] it is enough to show that the two functors are G -homotopy invariant, G -excisive, and homotopy equivalent when restricted to the orbit category $\mathbf{Or}(G)$. For the first two properties, we apply [5, 1.28, 4.2]. For the last property, we follow the method of [11, §8]. Let $T = G/H$ and consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A})_\emptyset & \longrightarrow & \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A}) & \longrightarrow & \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A})^{>0} \\ \simeq \downarrow F & & \downarrow F & & \downarrow F \\ \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))_\emptyset & \xrightarrow{\sim} & \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A})) & \xrightarrow{\sim} & \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))^{>0} \end{array}$$

where the vertical maps are induced by the additive functors of Lemma 2.2. We apply $\lambda = \mathbb{K}^\infty$ or $\lambda = \mathbb{L}^\infty$ to obtain fibrations of spectra. Note that λ applied to either of the middle two categories gives a spectrum with trivial homotopy groups (by an Eilenberg swindle). Therefore the first and third vertical maps induce a homotopy equivalence of spectra. Since the level-preserving condition is automatic on the germ categories, we are done. $\square$

The next property allows us to divide out a normal subgroup in suitable circumstances.

Lemma 2.5. *Let N be a normal subgroup of G , and $\mathcal{A}$ be an additive category with right G -action such that N acts trivially. Let X be a G -CW complex such that N acts freely on X . Then there is an additive functor*

$$\mathcal{D}^G(X \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}^{G/N}(N \backslash X \times [0, 1]; \mathcal{A})$$

which induces an isomorphism on K -theory after taking germs away from the empty set.

Proof. We will construct a functor $F = F_2 \circ F_1$ inducing this isomorphism in two steps. First, we have a functor $F_1: \mathcal{D}^G(X \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}^G(N \backslash X \times [0, 1]; \mathcal{A})$, which is the identity on objects and morphisms. The continuous control condition measured in X is stronger than the continuous control condition measured in $N \backslash X$, so this is well-defined. This functor induces a homotopy invariant and G -excisive functor

$$F_1: \mathcal{D}^G(G/H \times [0, 1]; \mathcal{A})^{>0} \rightarrow \mathcal{D}^G(N \backslash G/H \times [0, 1]; \mathcal{A})^{>0}$$

for $X = G/H$, and an equivalence $\mathcal{D}^G(G/H; \mathcal{A}) \simeq \mathcal{D}^G(N \backslash G/H; \mathcal{A})$. Therefore F_1 induces isomorphisms on K -theory after taking germs away from the empty set (as in the proof of Lemma 2.4). Secondly, there is a functor

$$F_2: \mathcal{D}^G(N \backslash X \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}^{G/N}(N \backslash X \times [0, 1]; \mathcal{A})$$

defined on objects by $F_2(A)_{(gN, \bar{y})} = A_{(g, \bar{y})}$, where $\bar{y} \in N \setminus X \times [0, 1)$. We define the functor on morphisms by $F_2(\phi)_{(gN, \bar{y})}^{(g'N, \bar{y}')} = \phi_{(g, \bar{y})}^{(g', \bar{y}')}$. This is well-defined by G -invariance of the objects and morphisms in the domain, and the continuous control conditions on morphisms agree since both are measured in $N \setminus X$. We also have an inverse functor F_2' defined by $F_2'(A)_{(e, \bar{y})} = A_{(eN, \bar{y})}$ on objects, extended by G -equivariance, and similarly for morphisms. It follows that F_2 is an equivalence of categories. $\square$

In the next statement, if $\mathcal{A}$ is an additive G -category, we denote by $\text{Res}_H \mathcal{A}$ the same category considered as an H -category under restriction to a subgroup H of G . The following is ‘‘Shapiro’s Lemma’’ in our setting.

Proposition 2.6. *Let H be a subgroup of G , $\mathcal{A}$ be an additive category with G -action, and X be an H -CW complex. There is an additive functor*

$$\mathcal{D}^H(X \times [0, 1); \text{Res}_H \mathcal{A}) \rightarrow \mathcal{D}^G(G \times_H X \times [0, 1); \mathcal{A})$$

which induces an equivalence of categories after taking germs.

Proof. This proposition is proven in [1, Proposition 8.3] in the case where $\mathcal{A}$ is the category of finitely generated free R -modules. The same proof works for any coefficient category once the functor $\text{Ind}: \mathcal{D}^H(X \times [0, 1); \text{Res}_H \mathcal{A}) \rightarrow \mathcal{D}^G(G \times_H X \times [0, 1); \mathcal{A})$ is defined for general $\mathcal{A}$. Let $\phi: A \rightarrow B$ be a morphism in $\mathcal{D}^H(X \times [0, 1); \text{Res}_H \mathcal{A})$. Then

$$\text{Ind}: \mathcal{D}^H(X \times [0, 1); \text{Res}_H \mathcal{A}) \rightarrow \mathcal{D}^G(G \times_H X \times [0, 1); \mathcal{A})$$

is defined by $\text{Ind}(A)_{[g, y]} = (g^{-1})^* A_y$, and $\text{Ind}(\phi)_{[g, y]}^{[g', y']} = (g^{-1})^* \phi_y^{g^{-1}g'y'}$ if $g^{-1}g' \in H$, and is zero otherwise. The inverse of this functor on the corresponding germ categories is induced by the inclusion $i: X \rightarrow G \times_H X$. That is, $\text{Ind}^{-1}(M)_y = M_{i(y)}$ and $\text{Ind}^{-1}(\psi)_y^{y'} = \psi_{i(y)}^{i(y')}$. $\square$

Remark 2.7. The equivalences given in these three properties are natural with respect to equivariant maps $X \rightarrow X'$. If $\mathcal{A}$ is an additive category with involution, one can check that the above properties continue to hold in this context. This is needed for applications to the L -theory assembly maps.

3. ASSEMBLY AND SUBGROUPS

The properties of the continuously controlled categories given so far lead to a formal statement about assembly and subgroups. This is just our version of [4, Proposition 4.2]. If H is a subgroup of G , and $\mathcal{A}$ is an additive H -category, we denote $\text{Ind}_H^G \mathcal{A} := \mathcal{D}_l^H(G; \mathcal{A})$ considered as a G -category by using the H - G biset structure of G .

Proposition 3.1. *Let $f: X \rightarrow X'$ be a G -equivariant map between G -CW complexes. Let H be a subgroup of G , and let $\mathcal{A}$ be an additive category with H -action. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{D}^H(\text{Res}_H X \times [0, 1); \mathcal{A})^{>\emptyset} & \xrightarrow{f_*} & \mathcal{D}^H(\text{Res}_H X' \times [0, 1); \mathcal{A})^{>\emptyset} \\ \uparrow \simeq & & \uparrow \simeq \\ \mathcal{D}^G(X \times [0, 1); \text{Ind}_H^G \mathcal{A})^{>\emptyset} & \xrightarrow{f_*} & \mathcal{D}^G(X' \times [0, 1); \text{Ind}_H^G \mathcal{A})^{>\emptyset} \end{array}$$

Proof. By Lemma 2.4 with $K = H$ and $S = G$, we have

$$\mathcal{D}^G(X \times [0, 1]; \text{Ind}_H^G \mathcal{A})^{>0} \simeq \mathcal{D}^{H \times G}(G \times X \times [0, 1]; \mathcal{A})^{>0}$$

where $1 \times G$ acts trivially on $\mathcal{A}$ in the right-hand side. Finally,

$$\mathcal{D}^{H \times G}(G \times X \times [0, 1]; \mathcal{A})^{>0} \simeq \mathcal{D}^H(\text{Res}_H X \times [0, 1]; \mathcal{A})^{>0}$$

by applying Lemma 2.5 to $H \times G$ with $N = G$. Note that G acts freely on $G \times X$, with quotient isomorphic to $\text{Res}_H X$. $\square$

Corollary 3.2. *Let H be a subgroup of G and $\mathcal{F}$ be a family of subgroups of G . Suppose that the K -theory or L -theory $(G, \mathcal{F}, \mathcal{B})$ -assembly map is an isomorphism (respectively injection or surjection) for every additive coefficient category $\mathcal{B}$ with G -action. Then the $(H, \mathcal{F}|_H, \mathcal{A})$ -assembly map is an isomorphism (respectively injection or surjection) for any additive coefficient category $\mathcal{A}$ with H -action.*

Proof. Just substitute $X = E_{\mathcal{F}}G$ and $X' = \bullet$ in the diagram above. $\square$

In particular, this says that the Farrell-Jones conjecture with coefficients is stable under taking subgroups. These ideas can be extended further to obtain a version of the fibered isomorphism conjecture.

Proposition 3.3. *Let $\phi: H \rightarrow G$ be a group homomorphism, and let $\mathcal{F}$ be a family of subgroups of G . If the K -theory or L -theory assembly map for G relative to the family $\mathcal{F}$ is an isomorphism (respectively injective or surjective), with twisted coefficients in any additive G -category, then the assembly map for H relative to the pull-back family $\phi^*\mathcal{F} = \{K \leq H \mid \phi(K) \in \mathcal{F}\}$ is an isomorphism (respectively injection or surjection), with twisted coefficients in any additive H -category.*

Proof. The proof is the same as for Proposition 3.1 using $X = E_{\mathcal{F}}G$ and $X' = \bullet$, with the action of H on $S = G$ and on X defined via ϕ , and $\text{Res}_{\phi} X = E_{\phi^*\mathcal{F}}G$. $\square$

4. ASSEMBLY FOR EXTENSIONS

In [12] the Baum-Connes conjecture for topological K -theory is shown to pass to extensions. We show that there is a similar statement for algebraic K - and L -theory.

The proof outline used in [12] has two main steps, which we now translate into our setting. In the first step we use a discrete transitive right G -set S , which can be expressed as a single orbit $S = \{s\} \cdot G$.

Proposition 4.1. *Let X be a G -CW complex, $S = \{s\} \cdot G$, and $\mathcal{A}$ be an additive G -category with involution. Then there is an additive functor*

$$\mathcal{D}^{G_s}(\text{Res}_{G_s} X \times [0, 1]; \text{Res}_{G_s} \mathcal{A}) \rightarrow \mathcal{D}^G(X \times [0, 1]; \mathcal{D}_I^0(S; \mathcal{A}))^{>0}$$

which induces a homotopy equivalence of spectra after applying $\mathbb{K}^{-\infty}$ or $\mathbb{L}^{-\infty}$. This equivalence is natural with respect to maps $X \rightarrow X'$ of G -CW complexes.

Proof. By Proposition 2.6,

$$\mathbb{K}^\infty(\mathcal{D}^{G_s}(\text{Res}_{G_s} X \times [0, 1]; \text{Res}_{G_s} \mathcal{A})^{>0}) \simeq \mathbb{K}^\infty(\mathcal{D}^G(G \times_{G_s} X \times [0, 1]; \mathcal{A})^{>0}).$$

Since $G \times_{G_s} X$ is G -equivariantly homeomorphic to $(G_s \backslash G) \times X = S \times X$, via the map $[g, x] \mapsto (Hg^{-1}, gx)$, and so

$$\mathcal{D}^G(G \times_{G_s} X \times [0, 1]; \mathcal{A})^{>0} \cong \mathcal{D}^G(S \times X \times [0, 1]; \mathcal{A})^{>0},$$

where $S \times X$ has the usual left G -action $g \cdot (s, x) = (sg^{-1}, gx)$. Finally, by Lemma 2.4,

$$\mathbb{K}^\infty(\mathcal{D}^G(S \times X \times [0, 1]; \mathcal{A})^{>0}) \simeq \mathbb{K}^\infty(\mathcal{D}^G(X \times [0, 1]; \mathcal{D}_l^0(S; \mathcal{A}))^{>0}).$$

The same proof works if we replace $\mathbb{K}^\infty$ by $\mathbb{L}^\infty$. $\square$

Example 4.2. Let $\pi: G \rightarrow K$ be a surjection of groups, and $V \subset K$ be a subgroup. We consider $S = K$ as a right- $(G \times V)$ -set via the transitive action $k \cdot (g, v) := \pi(g)^{-1}kv$, where $g \in G$, $v \in V$, and $k \in K$. Let X be a $(G \times K)$ -CW complex, and let $V' \subset G \times V$ denote the stabilizer subgroup of $e \in K$. Notice that $V' \cong \pi^{-1}(V)$, since $\pi(g)^{-1}v = e$ implies $g \in \pi^{-1}(v)$. By Proposition 4.1, we have a commutative diagram

$$\begin{array}{ccc} F_{V'}^\lambda(X; \mathcal{A}) & \longrightarrow & F_{V'}^\lambda(\bullet; \mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times V}^\lambda(X; \mathcal{D}_l^0(K; \mathcal{A})) & \longrightarrow & F_{G \times V}^\lambda(\bullet; \mathcal{D}_l^0(K; \mathcal{A})) \end{array}$$

for $\lambda = \mathbb{K}^\infty$ or $\lambda = \mathbb{L}^\infty$, which shows that the lower assembly map is a homotopy equivalence of spectra whenever the upper map is an equivalence.

Remark 4.3. In the proof of Theorem A, we will be using Example 4.2 with $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$, where $\mathcal{F}_G$ is a family of subgroups of G and $\mathcal{F}_K$ is a family of subgroups of K such that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$. If $V \in \mathcal{F}_K$, then the map $E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K \rightarrow E_{\mathcal{F}_G} G \times \bullet$ is a $G \times V$ -equivariant homotopy equivalence. Therefore, it is a V' -equivariant homotopy equivalence. Since $V' \cong \pi^{-1}(V)$, we have the homotopy commutative diagram:

$$\begin{array}{ccc} F_{\pi^{-1}(V)}^\lambda(E_{\mathcal{F}_G} G; \mathcal{A}) & \xrightarrow{a} & F_{\pi^{-1}(V)}^\lambda(\bullet; \mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times V}^\lambda(X; \mathcal{D}_l^0(K; \mathcal{A})) & \xrightarrow{b} & F_{G \times V}^\lambda(\bullet; \mathcal{D}_l^0(K; \mathcal{A})) \end{array}$$

where $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$.

If $V = K$, then $G \cong V' \subset G \times K$ and G acts on $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$ via this isomorphism. Since we are assuming that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$, X is a model for $E_{\mathcal{F}_G} G$. Thus, we have the homotopy commutative diagram:

$$\begin{array}{ccc} F_G^\lambda(E_{\mathcal{F}_G} G; \mathcal{A}) & \xrightarrow{c} & F_G^\lambda(\bullet; \mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times K}^\lambda(X; \mathcal{D}_l^0(K; \mathcal{A})) & \xrightarrow{d} & F_{G \times K}^\lambda(\bullet; \mathcal{D}_l^0(K; \mathcal{A})) \end{array}$$

Definition 4.4. Let G_1 and G_2 be discrete groups, and let X_1 and X_2 be G_1 - and G_2 -CW complexes, respectively. Let $\mathcal{A}$ be a $G_1 \times G_2$ -additive category with involution. The *partial assembly map*,

$$\mu^{G_1, G_2} : F_{G_1 \times G_2}^\lambda(X_1 \times X_2; \mathcal{A}) \rightarrow F_{G_2}^\lambda(X_2; \mathcal{D}^{G_1}(\bullet; \mathcal{A})),$$

is the map induced by the second factor projection $X_1 \times X_2 \rightarrow \bullet \times X_2$, composed with the homotopy equivalence from Lemma 2.4 with $S = \bullet$.

Lemma 4.5. *The partial assembly map is natural in the control spaces and involution invariant.* $\square$

Now the second step of the proof outline gives a criterion for the partial assembly map to be an equivalence.

Proposition 4.6. *Let G and K be groups, and let $\mathcal{B}$ be an additive $G \times K$ -category. Let $\mathcal{F}_K$ be a family of subgroups of K . Let X_1 be a G -CW complex and X_2 be a K -CW complex with isotropy in $\mathcal{F}_K$. Suppose that*

$$F_{G \times V}^\lambda(X_1 \times \bullet; \mathcal{B}) \rightarrow F_{G \times V}^\lambda(\bullet; \mathcal{B})$$

is a homotopy equivalence for all subgroups $V \in \mathcal{F}_K$. Then the partial assembly map

$$\mu^{G, K} : F_{G \times K}^\lambda(X_1 \times X_2; \mathcal{B}) \rightarrow F_K^\lambda(X_2; \mathcal{D}^G(\bullet; \mathcal{B}))$$

is also an equivalence for $\lambda = \mathbb{K}^{-\infty}$ or $\lambda = \mathbb{L}^{-\infty}$.

Proof. Suppose that $X_2 = K/V$ for some $V \in \mathcal{F}_K$. Then, by Shapiro's Lemma (Proposition 2.6),

$$\begin{array}{ccc} F_{G \times V}^\lambda(X_1 \times \bullet; \mathcal{B}) & \xrightarrow{\mu^{G, V}} & F_V^\lambda(\bullet; \mathcal{D}^G(\bullet; \mathcal{B})) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times K}^\lambda(X_1 \times K/V; \mathcal{B}) & \xrightarrow{\mu^{G, K}} & F_{G \times K}^\lambda(K/V; \mathcal{D}^G(\bullet; \mathcal{B})) \end{array}$$

and the upper map is an equivalence by assumption, since $F_V^\lambda(\bullet; \mathcal{D}^G(\bullet; \mathcal{B})) \simeq F_{G \times V}^\lambda(\bullet; \mathcal{B})$. The functors $H(X_2) := F_{G \times K}^\lambda(X_1 \times X_2; \mathcal{B})$ and $H'(X_2) := F_K^\lambda(X_2; \mathcal{D}^G(\bullet; \mathcal{B}))$ are homotopy-invariant and K -excisive functors from K -CW complexes to spectra. Since $H(K/V) \simeq H'(K/V)$ for all $V \in \mathcal{F}_K$, we conclude that $H(X_2) \simeq H'(X_2)$ for all K -CW complexes with isotropy in $\mathcal{F}_K$. $\square$

The following is our main result about extensions:

Theorem 4.7. *Let $N \rightarrow G \xrightarrow{\pi} K$ be a group extension, where $N \triangleleft G$ is a normal subgroup, and K is the quotient group. Let $\mathcal{F}_G$ be a family of subgroups of G and $\mathcal{A}$ an additive category with right G -action. Let $\mathcal{F}_K$ be a family of subgroups of K such that $\pi(H) \in \mathcal{F}_K$ for every $H \in \mathcal{F}_G$. Suppose that for every $V \in \mathcal{F}_K$ the $(\pi^{-1}(V), \mathcal{F}_G|_{\pi^{-1}(V)}, \mathcal{A})$ -assembly map in algebraic K -theory is an isomorphism, and that for every additive category $\mathcal{B}$ with right K -action the $(K, \mathcal{F}_K, \mathcal{B})$ -assembly map in algebraic K -theory is injective (resp. surjective). Then the $(G, \mathcal{F}_G, \mathcal{A})$ -assembly map in algebraic K -theory is injective (resp. surjective).*

The same statement holds for algebraic L -theory as well.

Example 4.8. Suppose that N is *finite* normal subgroup of G . Then the Farrell-Jones conjecture with twisted coefficients holds for G if it holds for $K = G/N$.

Example 4.9. Suppose that $1 \rightarrow N \rightarrow G \rightarrow K \rightarrow 1$ is a group extension, and $\mathcal{F}_G$ and $\mathcal{F}_K$ both denote the family of finite subgroups of their respective groups. Then the conclusions of Theorem 4.7 hold provided that the assembly map is injective (resp. surjective) for K and for every subgroup of G containing N as a subgroup of finite index.

The Proof of Theorem 4.7. Let $X = E_{\mathcal{F}_G} G \times E_{\mathcal{F}_K} K$. Let $V \in \mathcal{F}_K$ be given. By Remark 4.3, we have a homotopy commutative diagram:

$$\begin{array}{ccc} F_{\pi^{-1}(V)}^\lambda(E_{\mathcal{F}_G} G; \mathcal{A}) & \xrightarrow{a} & F_{\pi^{-1}(V)}^\lambda(\bullet; \mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times V}^\lambda(X; \mathcal{D}_l^0(K; \mathcal{A})) & \xrightarrow{b} & F_{G \times V}^\lambda(\bullet; \mathcal{D}_l^0(K; \mathcal{A})) \end{array}$$

Let $\mathcal{B} = \mathcal{D}_l^0(K; \mathcal{A})$, and note that the upper map a is an equivalence by assumption, since $\text{Res}_{\pi^{-1}(V)} E_{\mathcal{F}_G} G$ is a universal space for the family $\mathcal{F}_G|_{\pi^{-1}(V)}$. Hence, the lower map b is also an equivalence. By Proposition 4.6, we have the homotopy commutative diagram:

$$\begin{array}{ccc} F_{G \times K}^\lambda(X; \mathcal{B}) & \xrightarrow{d} & F_{G \times K}^\lambda(\bullet; \mathcal{B}) \\ \mu^{G,K} \downarrow \simeq & & \downarrow \simeq \\ F_K^\lambda(E_{\mathcal{F}_K} K; \mathcal{D}^G(\bullet; \mathcal{B})) & \xrightarrow{e} & F_K^\lambda(\bullet; \mathcal{D}^G(\bullet; \mathcal{B})) \end{array}$$

By assumption, the map e is injective (resp. surjective), which implies that d is injective (resp. surjective).

Using Remark 4.3 again, we have the homotopy commutative diagram:

$$\begin{array}{ccc} F_G^\lambda(E_{\mathcal{F}_G} G; \mathcal{A}) & \xrightarrow{c} & F_G^\lambda(\bullet; \mathcal{A}) \\ \downarrow \simeq & & \downarrow \simeq \\ F_{G \times K}^\lambda(X; \mathcal{D}_l^0(K; \mathcal{A})) & \xrightarrow{d} & F_{G \times K}^\lambda(\bullet; \mathcal{D}_l^0(K; \mathcal{A})) \end{array}$$

Therefore, the assembly map c is injective (resp. surjective). $\square$

Corollary 4.10. *The Farrell-Jones conjecture with twisted coefficients is true for $G_1 \times G_2$ if and only if it is true for G_1 , G_2 , and every product $V_1 \times V_2$, where $V_1 \leq G_1$ and $V_2 \leq G_2$ are virtually cyclic subgroups.*

Proof. By our main result applied to the projection $G_1 \times G_2 \rightarrow G_2$, we may assume that G_2 is virtually cyclic. Similarly, we may assume that G_1 is virtually cyclic. Thus, we are reduced to knowing the conjecture for products $V_1 \times V_2$ of virtually cyclic subgroups of G_1 and G_2 respectively. $\square$

Remark 4.11. A product $V_1 \times V_2$ of virtually cyclic subgroups can be further reduced to the basic cases $\mathbf{Z} \times \mathbf{Z}$, $\mathbf{Z} \times D_\infty$ and $D_\infty \times D_\infty$ after quotients by finite normal subgroups.

5. THE PROOF OF LEMMA 2.2

We will check the details of Lemma 2.2, which asserts that there is an additive functor

$$F: \mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A}) \rightarrow \mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))$$

defined by

$$\begin{aligned} (F(A)_{(g,t)})_{(k,s)} &:= A_{(k,g,s,t)} \\ \left(F(\phi)_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')} &:= \phi_{(k,g,s,t)}^{(k',g',s',t')}. \end{aligned}$$

Here $\mathcal{A}$ is an additive category with commuting right K and G -actions, T a left G -set and S a transitive K - G biset. The group $K \times G$ acts on $S \times T$ by the formula $(k, g) \cdot (s, t) := (ks g^{-1}, gt)$. Recall the notation (k, s) for elements of $K \times S$, and (g, t) for elements of $G \times T \times [0, 1]$. We will let $\epsilon: T \times [0, 1] \rightarrow T$ denote the projection map. Notice that $\phi_{(k,g,s,t)}^{(k',g',s',t')} = 0$ unless $s = s'$ and $\epsilon(t) = \epsilon(t')$, since the morphisms $\phi: A \rightarrow B$ in the domain category are assumed to be level-preserving. The free $(K \times G)$ -space

$$\overline{S} \times \overline{T} = K \times G \times S \times T \times [0, 1]$$

is a resolution for $S \times T \times [0, 1]$. The proof that F is a functor is done in the following steps.

(1). $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$. Since

$$\left(F(\phi) \circ F(\psi) \right)_{(g,t)}^{(g',t')} = \sum_{(g'',t'')} F(\phi)_{(g'',t'')}^{(g',t')} \circ F(\psi)_{(g,t)}^{(g'',t'')}$$

we have that:

$$\begin{aligned} \left((F(\phi) \circ F(\psi))_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')} &= \left(\sum_{(g'',t'')} F(\phi)_{(g'',t'')}^{(g',t')} \circ F(\psi)_{(g,t)}^{(g'',t'')} \right)_{(k,s)}^{(k',s')} \\ &= \sum_{(g'',t'')} \left(F(\phi)_{(g'',t'')}^{(g',t')} \circ F(\psi)_{(g,t)}^{(g'',t'')} \right)_{(k,s)}^{(k',s')} \\ &= \sum_{(g'',t'')} \sum_{(k'',s'')} \phi_{(k'',g'',s'',t'')}^{(k',g',s',t')} \circ \psi_{(k,g,s,t)}^{(k'',g'',s'',t'')} \\ &= (\phi \circ \psi)_{(k,g,s,t)}^{(k',g',s',t')} \\ &= \left(F(\phi \circ \psi)_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')} \end{aligned}$$

(2). $F(A)_{(g,t)}$ is an object of $\mathcal{D}_l^K(S; \mathcal{A})$, for every $(g, t) \in G \times T \times [0, 1]$.

(2'). $F(A)_{(g,t)}$ is K -invariant. For each $h \in K$,

$$\begin{aligned} (h^*(F(A)_{(g,t)}))_{(k,s)} &= h^*((F(A)_{(g,t)})_{(hk,hs)}) \\ &= h^*(A_{(hk,g,hs,t)}) \\ &= (h^*A)_{(k,g,s,t)} \\ &= A_{(k,g,s,t)} \\ &= (F(A)_{(g,t)})_{(k,s)} \end{aligned}$$

(2''). The support of $F(A)_{(g,t)}$ is K -compact in $K \times S$.

Since a discrete K -set is K -compact if and only if its image under the quotient map is finite, we need to show that $K \setminus \text{supp}(F(A)_{(g,t)})$ is finite. Let p be the projection map from $K \times G \times S \times T \times [0, 1]$ to $K \times G \times S \times T$, $M = p(\text{supp}(A))$, and $N = p(\text{supp}(A) \cap K \times \{g\} \times S \times \{t\}) \subset M$. Consider the following commutative diagram, in which $f(k', g', s', t') = (k', s'g')$, $m_g(k, s) = (k, sg^{-1})$, and the vertical arrows are quotient maps.

$$\begin{array}{ccccc} K \times G \times S \times T & \xrightarrow{f} & K \times S & \xrightarrow{m_g} & K \times S \\ \downarrow q_{K \times G} & & \downarrow q_K & & \downarrow q_K \\ (K \times G) \setminus (K \times G \times S \times T) & \xrightarrow{\bar{f}} & K \setminus (K \times S) & \xrightarrow{\bar{m}_g} & K \setminus (K \times S) \end{array}$$

Since M is discrete and $(K \times G)$ -compact, $q_{K \times G}(M)$ is finite. Since $N \subset M$, $q_{K \times G}(N)$ is also finite. Therefore, $(\bar{m}_g \circ \bar{f} \circ q_{K \times G})(N) = (q_K \circ m_g \circ f)(N) = q_K(\text{supp}(F(A)_{(g,t)}))$ is finite.

(3). $F(\phi)_{(g,t)}^{(g',t')}$ is a morphism of $\mathcal{D}_l^K(S; \mathcal{A})$, for every $(g, t), (g', t') \in G \times T \times [0, 1]$.

(3'). $F(\phi)_{(g,t)}^{(g',t')}$ is K -invariant. The proof is similar to the proof of (2').

(3''). Fix $(k, s) \in K \times S$. Then, the following set is finite:

$$P = \left\{ (k', s') \in K \times S \mid \left(F(\phi)_{(g,t)}^{(g',t')} \right)_{(k,s)}^{(k',s')} \neq 0 \text{ or } \left(F(\phi)_{(g,t)}^{(g',t')} \right)_{(k',s')}^{(k,s)} \neq 0 \right\}.$$

The sets $\left\{ (k', g', s', t') \in K \times G \times S \times T \times [0, 1] \mid \phi_{(k,g,s,t)}^{(k',g',s',t')} \neq 0 \right\}$ and $\left\{ (k', g, s', t) \in K \times G \times S \times T \times [0, 1] \mid \phi_{(k',g,s',t)}^{(k,g,s,t)} \neq 0 \right\}$ are finite and their union projects onto P .

(3'''). $F(\phi)_{(g,t)}^{(g',t')}$ is level preserving in S . This is because ϕ is level-preserving in $S \times T$.

(4). $F(A)$ is an object of $\mathcal{D}_l^G(T \times [0, 1]; \mathcal{D}_l^K(S; \mathcal{A}))$.

(4'). $F(A)$ is G -invariant. For each $\gamma \in G$,

$$\begin{aligned}
 (\gamma^*(F(A)))_{(g,t)} &= (\gamma^*(F(A)_{(\gamma g, \gamma t)}))_{(k,s)} \\
 &= \gamma^*((F(A)_{(\gamma g, \gamma t)}))_{(k, s\gamma^{-1})} \\
 &= \gamma^*(A_{(k, \gamma g, s\gamma^{-1}, \gamma t)}) \\
 &= (\gamma^*A)_{(k, g, s, t)} \\
 &= A_{(k, g, s, t)} \\
 &= (F(A))_{(g,t)}(k, s)
 \end{aligned}$$

(4''). The support of $F(A)$ is relatively G -compact in $G \times T \times [0, 1)$.

Let $p: K \times G \times S \times T \times [0, 1) \rightarrow G \times T \times [0, 1)$ be the projection map. Since $\text{supp}(A)$ is relatively $(K \times G)$ -compact and $p(\text{supp}(A)) = \text{supp}(F(A))$, $\text{supp}(F(A))$ is relatively G -compact in $G \times T \times [0, 1)$.

(4'''). The support of $F(A)$ is locally finite in $G \times T \times [0, 1)$.

Let $(g, t) \in \text{supp}(F(A))$ be given. We must find an open neighborhood $U \subset G \times T \times [0, 1)$ of (g, t) such that $U \cap \text{supp}(F(A)) = \{(g, t)\}$. Let

$$L = \{(k, s) \in K \times S \mid (k, g, s, t) \in \text{supp}(A)\}.$$

From (1''), we know that L is K -compact. That is, $L = K \cdot (K_0 \times S_0)$, where $K_0 \subset K$ and $S_0 \subset S$ are finite sets. Since $\text{supp}(A)$ is locally finite in $K \times G \times S \times T \times [0, 1)$, for each $(k_i, s_i) \in K_0 \times S_0$, there is a neighborhood $U_i \subset T \times [0, 1)$ of t , such that

$$(\{k_i\} \times \{g\} \times \{s_i\} \times U_i) \cap \text{supp}(A) = \{(k_i, g, s_i, t)\}.$$

Thus, for each $(k, s) \in L$, there is an i , such that

$$(\{k\} \times \{g\} \times \{s\} \times U_i) \cap \text{supp}(A) = \{(k, g, s, t)\}.$$

Therefore, if we let $U = \{g\} \times (\cap_i U_i)$, then $U \cap \text{supp}(F(A)) = \{(g, t)\}$.

(5). $F(\phi)$ is a morphism in $\mathcal{D}_l^G(T \times [0, 1); \mathcal{D}_l^K(S; \mathcal{A}))$.

(5'). $F(\phi)$ is G -invariant. The proof is similar to the proof of (3').

(5''). Fix $(g, t) \in G \times T \times [0, 1)$. Then, the following set is finite

$$\left\{ (g', t') \in G \times T \times [0, 1) \mid F(\phi)_{(g,t)}^{(g', t')} \neq 0 \text{ or } F(\phi)_{(g', t')}^{(g, t)} \neq 0 \right\}.$$

As we saw in (2''), $\text{supp}(A) \cap K \times \{g\} \times S \times \{t\}$ is K -compact. Therefore, it is contained in $K \cdot (K_0 \times \{g\} \times S_0 \times \{t\})$, for some finite subsets $K_0 \subset K$ and $S_0 \subset S$. Notice that by K -equivariance, $F(\phi)_{(g,t)}^{(g', t')} \neq 0$ if and only if there exists an $s_0 \in S_0$, $k_0 \in K_0$ and $k' \in K$ such that $\phi_{(k_0, g, s_0, t)}^{(k', g', s, t')} \neq 0$. But for each $k_0 \in K_0$ and each $s_0 \in S_0$, there are only finitely many $k' \in K$, $g' \in G$ and $t' \in T \times [0, 1)$ such that $\phi_{(k_0, g, s_0, t)}^{(k', g', s_0, t')} \neq 0$. Therefore, there are only finitely many $g' \in G$ and $t' \in T \times [0, 1)$ such that $F(\phi)_{(g,t)}^{(g', t')} \neq 0$. A similar argument shows that there are only finitely many $g' \in G$ and $t' \in T \times [0, 1)$ such that $F(\phi)_{(g', t')}^{(g, t)} \neq 0$.

(5'''). $F(\phi)$ is continuously controlled in $T \times [0, 1]$.

Let $\phi: A \rightarrow B$ be a morphism in $\mathcal{D}_l^{K \times G}(S \times T \times [0, 1]; \mathcal{A})$. Let $(x_0, 1) \in T \times [0, 1]$ and a G_{x_0} -invariant neighborhood $U \subset T \times [0, 1]$ of $(x_0, 1)$ be given. We must find a G_{x_0} -invariant neighborhood $V \subset T \times [0, 1]$ of $(x_0, 1)$, such that $F(\phi)_{(g,t)}^{(g',t')} = 0 = F(\phi)_{(g',t')}^{(g,t)}$ whenever $(g, t) \in G \times V$ and $(g', t') \notin G \times U$.

By definition, $\left(F(\phi)_{(g,t)}^{(g',t')}\right)_{(k,s)}^{(k',s)} = \phi_{(k,g,s,t)}^{(k',g',s,t')}$. Let $s_0 \in S$ with $K \cdot s_0 \cdot G = S$, and let $H \leq K \times G$ be the stabilizer subgroup of s_0 . We will identify $G \times T \times [0, 1]$ with the level $\{s_0\} \times G \times T \times [0, 1]$. Notice that the intersection of $\text{supp}(A)$ with $K \times G \times \{s_0\} \times T \times [0, 1]$ is contained in,

$$\bigcup_{(a,b) \in H} a \cdot K_0 \times b \cdot G_0 \times \{s_0\} \times b \cdot T_0 \times [0, 1],$$

where $K_0 \subset K$, $G_0 \subset G$ and $T_0 \subset T$ are finite sets. This holds since $\text{supp}(A)$ is relatively $(K \times G)$ -compact and any element of $(K \times G) - H$ will move s_0 to another level in S .

Suppose that $\phi_{(k,g,s,t)}^{(k',g',s,t')} \neq 0$ for some $k \in K$, $g \in G$ and $t \in U$. Then we can write $\tau s \gamma^{-1} = s_0$, for some $\tau \in K$ and some $\gamma \in G$. By equivariance, $\phi_{(\tau k, \gamma g, s_0, \gamma t)}^{(\tau k', \gamma g', s_0, \gamma t')} \neq 0$. For this to happen, $(\tau k, \gamma g, s_0, \gamma t) \in \text{supp}(A)$. This implies that there exists $(a, b) \in H$ such that

$$(\tau k, \gamma g, s_0, \gamma t) \in a \cdot K_0 \times b \cdot G_0 \times \{s_0\} \times b \cdot T_0 \times [0, 1],$$

which is equivalent to saying that

$$(a^{-1} \tau k, b^{-1} \gamma g, s_0, b^{-1} \gamma t) \in K_0 \times G_0 \times \{s_0\} \times T_0 \times [0, 1]$$

In particular, $b^{-1} \gamma t \in b^{-1} \gamma U \cap (T_0 \times [0, 1])$.

Since T_0 is finite, there are only finitely many elements of G , say $\{g_1, g_2, \dots, g_r\}$, such that $g_i U \cap (T_0 \times [0, 1]) \neq \emptyset$. Therefore, $\gamma = b g_i$ for some $(a, b) \in H$ that fixes s_0 and some i with $1 \leq i \leq r$.

Since ϕ is continuously controlled at $g_i \cdot (x_0, 1)$ along $S \times T \times 1$, there is a neighborhood $V_i \subset T \times [0, 1]$ of $(x_0, 1)$ such that $\phi_{(k,g,s_0,g_i t)}^{(k',g',s_0,g_i t')} = 0$ if $t \in V_i$ and $t' \notin U$, for $1 \leq i \leq r$.

Let $V = \cap_i V_i$. Then, if $t \in V$ and $t' \notin U$, we have

$$\phi_{(a^{-1} \tau k, g_i g, s_0, g_i t)}^{(a^{-1} \tau k', g_i g', s_0, g_i t')} = 0$$

and hence

$$0 = \phi_{(\tau k, b g_i g, s_0, b g_i t)}^{(\tau k', b g_i g', s_0, b g_i t')} = \phi_{(\tau k, \gamma g, s_0, \gamma t)}^{(\tau k', \gamma g', s_0, \gamma t')} = \phi_{(k, g, s, t)}^{(k', g', s, t')}$$

by equivariance of the morphisms, and the relations $\gamma = b g_i$, $s_0 = \tau s \gamma^{-1}$. A similar argument shows that $F(\phi)_{(g',t')}^{(g,t)} = 0$ if $t \in V$ and $t' \notin U$. Therefore $F(\phi)$ is continuously controlled along $T \times 1$. $\square$

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