ON THE ASYMPTOTIC DIMENSION OF THE DUAL GROUP OF A LOCALLY COMPACT ABELIAN GROUP

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ABSTRACT. We show that the covering dimension of a locally compact abelian group coincides with the asymptotic dimension of its Pontryagin dual.

The goal of this paper is to show that the covering dimension of a locally compact abelian group coincides with the asymptotic dimension of its Pontryagin dual (see Theorem 4). This result supports the intuition that the Fourier transform should interchange the roles of small scales in a locally compact abelian group with large scales in its dual group.

Let T denote the multiplicative topological group of complex numbers with unit modulus. Let G be an abelian topological group (note that in this paper all topological groups under consideration will be Hausdorff). The *dual group* of G, denoted by G^*, is the set of continuous homomorphisms from G into T with multiplication defined pointwise and endowed with the compact-open topology. A *torus* is the topological product of (perhaps infinitely many) T's. We write $\mathbb{R}$ for the additive topological group of real numbers and $\mathbb{Z}$ for the integers as a discrete group.

For a topological space X, let $\dim(X)$ denote its covering dimension and $\mathrm{ind}(X)$ its small inductive dimension. Pasynkov showed [7] that if G is a locally compact topological group then $\dim(G) = \mathrm{ind}(G)$ and so for the purpose of this paper either notion of topological dimension suffices.

Gromov, [2], introduced the notion of the *asymptotic dimension* of a metric space as a means of studying the large scale geometry of groups. Generalizing beyond metric spaces, Roe, [8], defined asymptotic dimension for a *coarse space*, that is, a set X together with a collection, called a *coarse structure*, of subsets of $X \times X$ satisfying certain axioms. In [6], we showed that a topological group G has an intrinsic coarse structure, which we called

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the *group-compact coarse structure*, given by

$$\{E \subset G \times G \mid \text{there exists a compact set } A \subset G \text{ such that } E \subset G(A \times A)\}.$$

Also in [6], we defined $\mathrm{asdim}(G)$, the *asymptotic dimension of* G, to be the asymptotic dimension of G equipped with the group-compact coarse structure. This is equivalent to the following definition.

Definition 1. (See [6, Proposition 3.3].) Let G be a topological group and n a non-negative integer. Then $\mathrm{asdim}(G) \leq n$ if for every compact $K \subset G$ there exists a cover $\mathcal{U}$ of G such that:

(1) $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$,
(2) for each index i, and for each pair of distinct $A, B \in \mathcal{U}_i$ we have $(B^{-1}A) \cap K = \emptyset$,
(3) there exists a compact set L (depending on K) such that $U^{-1}U \subset L$ for every $U \in \mathcal{U}$.

If no such such integer n exists we say $\mathrm{asdim}(G) = \infty$. Also, $\mathrm{asdim}(G) = n$ if $\mathrm{asdim}(G) \leq n$ and the assertion $\mathrm{asdim}(G) \leq n - 1$ is false.

A topological group G is *compactly generated* if there exists a compact subset of G that generates G algebraically as an abstract group. If G is locally compact and compactly generated, with compact generating set Σ, then the group-compact coarse structure on G coincides with the bounded coarse structure for the word metric d_Σ on G corresponding to the generating set Σ ([6, Proposition 2.18]). Hence for such groups $\mathrm{asdim}(G)$, as given by Definition 1, coincides with the asymptotic dimension of the metric space (G, d_Σ). For an arbitrary discrete group G, [6, Corollary 3.14] implies that

$$\mathrm{asdim}(G) = \sup\{\mathrm{asdim}(H) \mid H \text{ is a finitely generated subgroup of } G\}$$

and so our definition coincides with that of [1] for discrete groups.

In order to prove our main theorem (Theorem 4), we first consider the case of a compact abelian group.

Lemma 2. *Let G be a compact abelian group. Then* $\dim(G) = \mathrm{asdim}(G^*)$.

Proof. Note that G^* is discrete since G is compact [4, Theorem 12]. By [1, Theorem 3.2], $\mathrm{asdim}(G^*) = \dim_{\mathbb{Q}}(G^* \otimes \mathbb{Q})$ where $\mathbb{Q}$ is the field of rational numbers and $\dim_{\mathbb{Q}}(V)$ denotes the vector space dimension of a $\mathbb{Q}$-vector space V.

It is a consequence of [3, Corollary 8.18] that there exists a compact subgroup D of G such that G/D is a (perhaps infinite dimensional) torus and $\dim(D) = 0$. We have $\dim(G) = \dim(D) + \dim(G/D)$ by [5, Theorem 2.1] and so $\dim(G) = \dim(G/D)$.

In the case $\dim(G) = \dim(G/D) = n < \infty$, [3, Theorem 8.22(5)] implies that $n = \dim_{\mathbb{Q}}(G^* \otimes \mathbb{Q})$ and so $\dim(G) = \operatorname{asdim}(G^*)$.

In the case $\dim(G) = \dim(G/D) = \infty$, the group G^* contains a subgroup isomorphic to $(G/D)^*$, and hence free abelian subgroups of arbitrarily large finite rank from which it follows that $\operatorname{asdim}(G^*) = \infty$. □

Next, we consider the case of a compactly generated locally compact abelian group.

Lemma 3. *Let G be a compactly generated locally compact abelian group. Then* $\dim(G) = \operatorname{asdim}(G^*)$.

Proof. By [4, Theorem 24], G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times K$ where a and b are non-negative integers and K is a compact group. Hence G^* is topologically isomorphic to $\mathbb{R}^a \times T^b \times K^*$ because the dual of a finite product is the product of the duals [4, Theorem 13]. Covering dimension is additive for a finite product of locally compact groups ([5, Theorem 2.1]) and so $\dim(G) = a + \dim(K)$. The closed subgroup $\mathbb{Z}^a \times (0) \times K^* \subset \mathbb{R}^a \times T^b \times K^*$ has compact quotient (topologically isomorphic to T^{a+b}) and so the inclusion $\mathbb{Z}^a \times (0) \times K^* \hookrightarrow \mathbb{R}^a \times T^b \times K^*$ is a coarse equivalence by [6, Proposition 2.34]. It follows that $\operatorname{asdim}(\mathbb{Z}^a \times K^*) = \operatorname{asdim}(G^*)$. Note that $\mathbb{Z}^a \times K^*$ is a discrete abelian group and so by [1, Corollary 3.3] we have $\operatorname{asdim}(\mathbb{Z}^a \times K^*) = \operatorname{asdim}(\mathbb{Z}^a) + \operatorname{asdim}(K^*) = a + \operatorname{asdim}(K^*)$. Lemma 2 asserts that $\dim(K) = \operatorname{asdim}(K^*)$. Hence $\dim(G) = a + \dim(K) = a + \operatorname{asdim}(K^*) = \operatorname{asdim}(G^*)$. □

Theorem 4. *Let G be a locally compact abelian group. Then* $\dim(G) = \operatorname{asdim}(G^*)$.

Proof. By [4, Corollary 2 of Theorem 21] G has an open subgroup U that is compactly generated. Since U is open G/U is discrete and thus $\dim(G/U) = 0$. Hence $\dim(G) = \dim(U) + \dim(G/U) = \dim(U)$ (again using [5, Theorem 2.1]). Let $i\colon U \to G$ be the inclusion and $p\colon G \to G/U$ the quotient map. By [4, Proposition 36],

$$0 \to (G/U)^* \xrightarrow{p^*} G^* \xrightarrow{i^*} U^* \to 0$$

is an exact sequence; furthermore, p^* is a homeomorphism onto its image and i^* is an open map. Hence U^* is the quotient of G^* by the image of p^*. Note that $(G/U)^*$ is compact since G/U is discrete [4, Theorem 12]. By [6, Proposition 2.35], i^* is a coarse equivalence and thus $\operatorname{asdim}(G^*) = \operatorname{asdim}(U^*)$. Since U is compactly generated, Lemma 3 gives that $\dim(U) = \operatorname{asdim}(U^*)$. Hence $\dim(G) = \dim(U) = \operatorname{asdim}(U^*) = \operatorname{asdim}(G^*)$. □

Applying Theorem 4 to G^* and using the duality theorem of Pontryagin and Van Kampen, that is, the topological isomorphism between G and G^{**}, immediately yields the following corollary.

Corollary 5. *Let G be a locally compact abelian group. Then* $\dim(G^*) = \operatorname{asdim}(G)$.

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