HOW MANY HOLES DOES MY CAT HAVE?

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In memory of Professor Bill Watson

Abstract. Euler's famous formula $V - E + F = 2$ is one of the most elegant and powerful formulas in mathematics, and for this reason it was exactly the sort of topic that the late Professor Bill Watson used to engage students and to deliver popular Math Club lectures. In this paper we will discuss Euler's formula, a few of its consequences, and, ultimately, Professor Watson's unique answer to the title of this paper.

Introduction

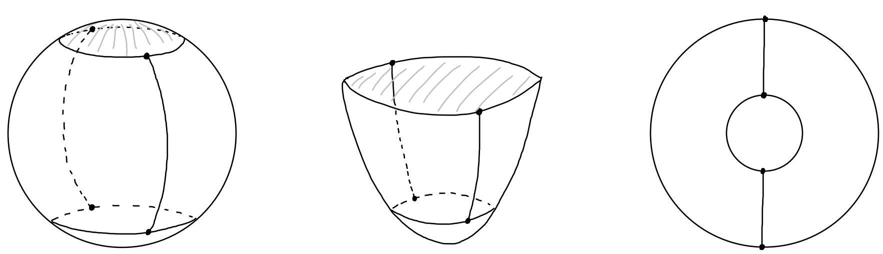
"How many holes does my cat have?" That was the title of a lecture that my late colleague Bill Watson gave to the St. John's University Math Club every several years. The title always caused a bit of a ruckus. Students and professors alike could be heard debating the anatomy of a cat in the department hallway for the entire week leading up to his lecture. Bill enjoyed listening to these (sometimes gruesome) conversations, but he never let the cat out of the bag, so to speak. He simply said that if you wanted to know the answer to his question, to hear the punch line to his joke, then you had to attend his lecture. And the turnout was always great. For those of us who knew Bill, theatrics like this were his style. He was an energetic guy with a wide variety of interests ranging from Quantum Physics to Biochemistry to African art to translating Mayan hieroglyphics. But his first love was mathematics, beautiful and powerful mathematics. He was a differential geometer, best known for his work on almost Hermitian submersions and the Goldberg conjecture [Wa76, Wa00]. Bill loved to teach, and was very good at it. He was awarded the prestigious St. Vincent de Paul Teacher-Scholar Award at St. John's University in 1998 and had a loyal following of students. Bill enjoyed talking about mathematics, and about life, and he lived for what he called the "Oh Wow!" factor. By that he meant the look of exhilaration on a student's face when they suddenly understood a deep and beautiful piece of mathematics. Bill believed in using powerful theorems to solve problems and he taught

Date: April 12, 2012.

his students to do the same. For example, his first year calculus classes were exposed to ideas from Morse theory, which they used to graph functions. He had a flamboyant style of teaching, as illustrated by his infamous Math Club lecture. In this short article, I will try to capture the essence of his talk and do my best to deliver the punch line of his joke.

The Lecture

The story begins with one of the most beautiful formulas in all of mathematics, a formula known as the *Euler characteristic of the sphere*. Suppose that you partition a sphere into finitely many polygonal-like pieces, called *faces*, each of which has a boundary drawn with finitely many *vertices* (i.e., points), and *edges* (i.e., paths between vertices that do not intersect each other and have no self-intersections). Figure 1(a) shows one such partition. These requirements are rather loose. For example, choosing a single vertex and a single edge beginning and ending at that vertex (i.e., a loop like the equator), yields a partition of the sphere that has two faces (like the two hemispheres divided by the equator). Now let V be the number of vertices, E be the number of edges and F be the number of faces in your partition. Then no matter how you decompose the sphere you will get $V - E + F = 2$. How can this always be true? Here is a nice short proof that Bill liked [Ri08, Chapter 13].



(a) Remove the top face. (b) Stretch out the hole. (c) Flatten onto the plane.

Figure 1. The Euler characteristic of a sphere.

Start with any decomposition of the sphere into V vertices, E edges and F faces. Then pick one face and remove it. Now imagine that what is left of the sphere is made of pliable material and stretch it out so that it is completely flat (see Figure 1). What we then have is a flattened surface decomposed into V vertices, E edges, and $F - 1$ faces[1]. Thus, its

[1] We do not count the unbounded region as a face.

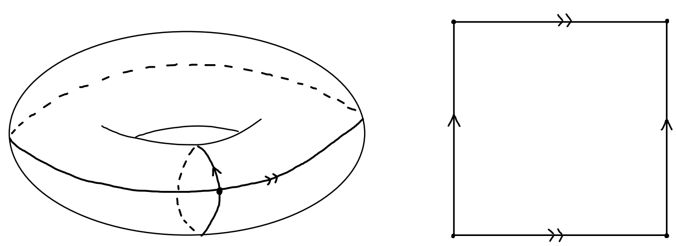
Euler expression, vertices minus edges plus faces, is $V - E + (F - 1)$, which is equal to $(V - E + F) - 1$, one less than the Euler expression for the given decomposition of the sphere. So if we show that the Euler number of the flattened surface is 1, then we will have shown that $V - E + F = 2$ for the sphere. To achieve this we will systematically "kill" the edges of the flattened surface.

The method for "killing" an edge is as follows. Pick an edge. The edge has either one or two vertices. If the edge connects two vertices, then we can shrink it to a point to obtain a new flat region in the plane. The process of shrinking the edge reduces the number of edges by one and the number of vertices by one. Therefore the value of the Euler expression, vertices minus edges plus faces, is unchanged. If the edge has only one vertex, then it is a loop and we simply remove it (but not its vertex). Since a loop encloses a face, this has the effect of decreasing the number of edges by one and the number of faces by one. Again, the Euler number is unchanged. If we continually "kill" edges in this way, then we will eventually be left with a single vertex, and its Euler expression, vertices minus edges plus faces, is clearly 1 because it has no edges and no faces. Therefore, we have shown that the Euler number of our flattened surface is 1 and, hence, $V - E + F = 2$ for the sphere. □

This amazing formula for the sphere was discovered by Euler in 1750 (although not quite in the form stated here) while he was studying polyhedra. He used it to give an elementary proof that there are only five Platonic solids; i.e., convex regular polyhedra. His formula has many interesting applications. For example, a standard soccer ball is made of regular pentagons and regular hexagons. Using Euler's formula for the sphere and some high school algebra, it is an easy exercise to show that one must use 12 pentagons to construct such a ball. This type of structure also appears in nature. A *fullerene* is a molecule made out of only carbon atoms. Since spherical fullerenes are composed of regular pentagons and regular hexagons, the same exercise shows that every spherical fullerene has exactly 12 pentagons. (There are many more neat applications of Euler's formula. See [Ri08] for a very nice book devoted to this topic written for a general audience.)

But Euler's formula has many more applications than just to polyhedra. In fact, it gave birth to a new subject now known as *topology*. Topology is the study of those properties of spaces that are unaffected by shrinking or twisting or other continuous deformations. The standard joke is that topologists cannot tell the difference between a coffee cup and a doughnut, since if we imagine the coffee cup to be made of soft clay, then we can mush the cup into the shape of a doughnut with the handle becoming the doughnut hole. The point is that we can do this without ripping or tearing the clay. Notice that if we continuously deform a sphere, like stretching it into the shape of a football, then no matter what we do the formula $V - E + F$ remains equal to 2. In fact, our proof made use of this kind of flexibility. But what about more general surfaces? Does Euler's expression always make sense? In fact it does, but this requires some "heavy lifting" as Bill would say. It is not completely obvious that any surface can be decomposed into vertices, edges and faces and that $V - E + F$ does not depend on this choice. Nevertheless, it is true that for every surface S the Euler characteristic of S, written as $\chi(S) = V - E + F$, is well-defined. Furthermore, the Euler characteristic is unchanged by shrinking or twisting or other continuous deformations, and for this reason is called a *topological invariant*.[2]

Let us consider another familiar space, the surface of a doughnut, also known as the *torus*. Can we continuously deform it into a sphere? Intuitively the answer is no. The torus has a hole in it and the sphere does not. But how can we turn this idea into a proof? We can use Euler's formula. If it were possible to deform the torus into a sphere, then we would have to get $V - E + F = 2$ for any decomposition of the torus into vertices, edges and faces. Consider the decomposition in Figure 2(a). It has one vertex, two edges and one face. (Figure 2(b) gives an alternative view of the torus and the chosen decomposition, where opposite edges are glued together to get the first picture.) With this configuration, $V - E + F = 0$, therefore the torus cannot be deformed into a sphere.



(a) (b)

FIGURE 2. The torus.

[2]Technically, a topological invariant is a property of a space that is unchanged by a *homeomorphism*, i.e., a continuous bijection whose inverse is also continuous.

There is another way to approach this calculation that turns out to be very useful. If we take any surface S, then we can "attach a handle" to it and construct a new surface T by cutting away two disjoint faces, and then gluing the boundaries of the removed faces to the ends of a *handle* (see Figure 4(a)). Topologically speaking, a handle is just a cylinder. As you might expect, we can calculate the Euler characteristic of the new surface, T, from the Euler characteristics of the original surface, S, and the cylinder. The Euler characteristic of the cylinder is easy to calculate from the sphere's because a handle can be deformed into a sphere with two disjoint faces removed (see Figure 3). Taking two faces away from

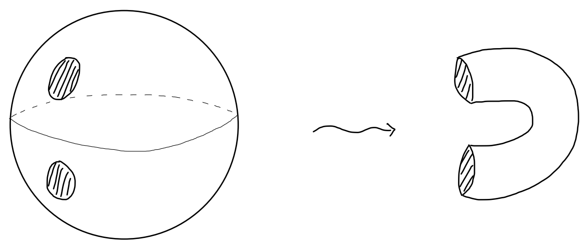
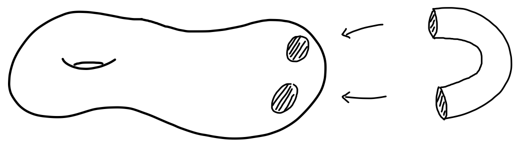


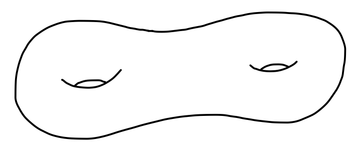
FIGURE 3. A handle.

the sphere reduces the Euler characteristic by two, therefore the Euler characteristic of a handle is 0. Now if we know the Euler characteristic of S, $\chi(S)$, then we can compute the Euler characteristic of T, $\chi(T)$, by understanding what happens to $\chi(S)$ when we attach a handle to S. As we saw with the sphere, cutting out two faces of S reduces its Euler characteristic by two, and since the Euler number of the handle is zero, the sum of the Euler numbers of the two disjoint pieces to be glued together is simply $\chi(S) - 2$. Now we need to see what happens to this number when the two pieces are glued together to make the surface T. Starting with one end of the handle, we take the circle from that end and identify it with the boundary of one of the removed faces in S. Before doing this we add vertices and edges to the circles, if necessary, to make sure that the vertices and edges of each of the circles coincide in a one to one fashion. When they get glued together, those vertices and edges are identified, and are therefore counted only once each. This seemingly affects the number $\chi(S) - 2$ that we computed above. However, no matter how a circle is divided up into vertices and edges, the number of vertices must always be equal to the number of edges. Therefore the gluing of these circles does not alter the total Euler number. Similarly, gluing in the other end of the handle also does not change the Euler number.

Therefore, $\chi(T) = \chi(S) - 2$. We can use this method to calculate the Euler characteristic of the torus a second way, since the torus can be constructed by attaching a handle to the sphere (like making a coffee cup out of clay). Thus, $\chi(\text{Torus}) = \chi(\text{Sphere}) - 2 = 2 - 2 = 0$.



(a)



(b)

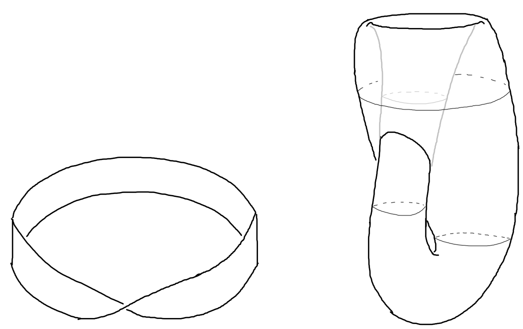
FIGURE 4. Attaching a handle to a torus to make a doughnut with two holes.

Now let's use this approach to analyze a doughnut with two holes. This surface is obtained by attaching a handle to the torus, as in Figure 4. Therefore its Euler characteristic is $0 - 2 = -2$ and we see an interesting pattern developing. If we start with a sphere and attach g handles, then we end up with a surface that looks like a doughnut with g holes, and we can compute its Euler characteristic by starting with the Euler characteristic of the sphere and subtracting 2 for every handle that we attached. That is, the Euler characteristic of a doughnut with g holes is $2 - 2g$. Since the Euler characteristic is a topological invariant, we now know that doughnuts with different numbers of holes are topologically distinct surfaces.

Knowing that the Euler characteristic is a topological invariant allows us to say with absolute certainty that if two surfaces have different Euler characteristics, then they are topologically distinct. But what can we say if two surfaces have the same Euler characteristic? Surprisingly it tells us a lot.

Theorem (Classification of Closed Surfaces)**.** *A closed surface is completely determined by its Euler characteristic and its orientability.*

Two of the terms in this theorem need to be explained. A *closed* surface is a compact surface that does not have a boundary. The sphere and the torus, as well as the doughnuts with more than one hole, are examples of closed surfaces, but a cylinder is not closed since it has a boundary; namely, the circles at its ends. Intuitively, a surface is *orientable* if it is impossible for a two-dimensional figure that is drawn on the surface to move around in a continuous way such that the figure's left and righthand sides get reversed. All of the examples we considered above are orientable. But if we draw a cartoon character on a *Möbius strip* (Figure 5(a)), then that character's left and righthand sides get switched as it walks all the way around the strip. The Möbius strip is not a closed surface since it has a circle for a boundary. An example of a closed non-orientable surface is a *Klein bottle* (Figure 5(b)). One way to understand orientability is the fact that the closed orientable surfaces are the ones you can realize in three dimensions, and the closed non-orientable ones can only be viewed in three dimensions if they pass through themselves, like the Klein bottle does. To fully realize a closed non-orientable surface, four dimensions are needed.



(a) The Möbius strip. (b) The Klein bottle.

Figure 5

The Classification Theorem tells us that if we have a closed orientable surface, then it must be a sphere with g handles, for some $g \geq 0$. In other words, it must be a doughnut with g holes, and the Euler characteristic keeps track of this.

The Punch Line

Now we come to the punch line. We have seen how the Euler characteristic distinguishes between closed orientable surfaces by counting the number of holes they have. We also know that a closed orientable surface is obtained from a sphere by attaching some number of handles. Working backwards, if we start with a surface with a certain number of holes, then we can "kill" the handles (i.e., remove them) one by one until we are left with a sphere. Occasionally mathematicians use colorful language to describe what they are doing. The process of removing handles from a surface is known as "surgery", and the mathematicians who do this are sometimes called “surgeons”. Of course mathematical surgeons are not so effective medically speaking, since their goal is to “operate” on their patient, the given surface, “killing” all of the handles until it is a sphere, at which point the surface is completely “dead” (since it has no more holes). If we consider a cat as a closed orientable surface, then the number of handles we have to “kill” in order for the cat to be completely “dead” is the same as the number of holes that it has. Since we all know that a cat has nine lives, it must also have nine holes.

Now a member of Bill’s audience might have objected to this answer. Certainly, one could argue that a cat should be considered as a surface with boundary, not without. Some of the audience might even have groaned, as was often the response to one of Bill’s patented lengthy plays on words. But no matter what the reaction to Bill’s joke was, he succeeded in his goal: the delivery of a memorable lecture about something absolutely beautiful. The Euler characteristic is a simple, yet powerful invariant for understanding surfaces, and is precisely the kind of mathematics that Bill loved to teach. His unique style of telling stories, mathematical or otherwise, always kept the attention of his listeners, and his love for life and for learning was infectious. Bill was a good friend and will be dearly missed.

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