Corrections and updates to my book “Metric Embeddings: bilipschitz and coarse embeddings into Banach spaces”

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February 14, 2018

Contents

Acknowledgement

I would like to that Miroslav Bacak for a list of corrections.

1 Johnson-Lindenstrauss Lemma

There are new achievements on the exactness of the Johnson-Lindenstrauss lemma, see [LN16+] and [AK16+].

2 Fibred embeddings

On page 32 I used *fibered* instead of *fibred*.

3 Definitions of type and cotype

In this book *type p* means *Rademacher type p* and *cotype q* means *Rademacher cotype q*.

4 To section 3.3

page 92, lines 7 and 8 from above: $i = k$ should be replaced by $k = 1$

5 To section 3.4 and Exercise 3.37: On the number of relevant scales for a finite metric space

For an n-element metric space $X = \{x_1, \ldots, x_n\}$ with metric d we introduce an array containing $\frac{n(n-1)}{2}$ numbers $d(x_i, x_j)$, $i < j$. We assume that all distances are at least 1. The number of *relevant scales* for the metric space X is defined as the number of intervals $[2^{i-1}, 2^i)$ $i = 1, 2, \ldots$ containing some elements of the array $d(x_i, x_j)$, $i < j$. Let $RS(n)$ be the maximal number of relevant scales which an n-element metric space may have.

Using the hint to Exercise 3.37 (or without it) one can construct a n-element metric space with $2n - 3$ relevant scales. Somewhat surprisingly one cannot get more, that is

$$RS(n) = 2n - 3.$$

The upper bound is due to Dömötör Pálvölgyi (posted on `MathOverflow.net` on July 28, 2013, see `http://mathoverflow.net/a/137976/955`). Here is his proof:

Represent the metric space as a weighted complete graph whose vertex set is X and the weights on its edges are the distances, $d(x_i, x_j)$. Consider a minimum weight spanning tree T in this graph. Denote the weights of the edges of this tree by $d_1, \ldots, d_{n-1}$, suppose $d_1 \leq \ldots \leq d_{n-1}$.

Lemma 5.1. *All distances of the metric space X are contained in the intervals of the form $[2^m, 2^{m+1})$ containing at least one number of the form $\sum_{i \in I} d_i$, where $I \subset \{1, \ldots, n\}$.*

Proof. In fact, for an edge uv with weight d, take the path in T between u and v. Let $d_{i_1} \leq \cdots \leq d_{i_k}$ be the weights of edges of this path. Then $d_{i_k} \leq d$, otherwise replacing the edge of weight d_{i_k} with uv would give a spanning tree with a smaller weight. Obviously $d \leq \sum_{j=1}^{k} d_{i_k}$. Now, if we consider binary intervals containing numbers $d_{i_k}, d_{i_k} + d_{i_{k-1}}, \ldots, d_{i_k} + d_{i_{k-1}} + \cdots + d_1$, we see that this is a sequence of consecutive binary intervals. In fact, no interval is missed because each next number is $\leq$ twice the previous. $\square$

Lemma 5.2. *Denote by $RS'(n)$ the maximum possible number of intervals of the form $[2^m, 2^{m+1})$ that contain at least one number of the form $x + \sum_{i \in I} d_i$ where $0 \leq x \leq d_1$ and $\emptyset \neq I \subset \{1, \ldots, n\}$ and the maximum is taken over all numbers $\{d_1, \ldots, d_n\}$ satisfying $0 \leq d_1 \leq \ldots \leq d_n$. Then $RS'(n) \leq 2n$.*

Proof. We use induction. If for every j we have $d_j \leq d_1 + \sum_{i<j} d_i$, then we have $d_j \leq 2^{j-1} d_1$ and thus $\sum_j d_j \leq (2^n - 1)d_1$, so all the numbers of the form $x + \sum_{i \in I} d_i$ are between d_1 and $2^n d_1$, in this case $RS'(n) \leq n+1$. On the other hand, if there is a j for which $d_j > d_1 + \sum_{i<j} d_i$, then divide the d_i numbers into two groups, depending on whether their index is less than j or not. For those d_i whose index is less than j, the induction hypothesis implies that sums of their subcollections are present in at most $2(j-1)$ binary intervals. For those whose index is at least j we apply this induction again, but now with d_j in the role of d_1. Since $d_j > d_1 + \sum_{i<j} d_i$ we get that the combined sums (we mean sums of some subcolletion in $\{d_1, \ldots, d_{j-1}\}$ and some subcolletion in $\{d_j, \ldots, d_n\}$ are present in at most $2n$ intervals, and we are done. $\square$

Lemma 5.1 implies that $RS(n) \leq RS'(n-1)$, therefore $RS(n) \leq 2n - 2$. To get $2n - 3$ we observe that when applying Lemma 5.2 to count the intervals for Lemma 5.1 we do not need to use x for the smallest group of terms; x is needed only for all further groups to compensate for sums of terms from smaller groups. This decreases the total by 1 and completes the proof.

6 To Section 4.2, page 108

Since we use k to denote the degrees of the graph (it is assumed to be k-regular), it is slightly confusing that the number of vertices in the graph is assumed to be $2k+1$ (of course, there is no relation between these numbers).

7 Page 114_8

S should be s

8 Page 115, formula (4.20)

In the mentioned formula $s_{uv}\langle f(u), f(v)\rangle$ should be replaced by $2s_{uv}\langle f(u), f(v)\rangle$ in each of the sides.

9 Page 117^4

d should be n

10 Proof of Theorem 4.28, end of the first paragraph

Replace "cardinality of $\mathbb{F}_2^n$" by "(cardinality of $\mathbb{F}_2^n$) $- 2^k$, because we need $x \notin V$".

11 End of the Proof of Theorem 4.28

Replace "exponent in (4.40) is $< n$" by "number in (4.40) is $< 2^n - 2^k$",

12 Section 4.5, equation (4.36)

In the equation and one line above it ℓ and l mean the same.

13 Chapter 5: Books on expanders

Tao [Tao15] has already published the book "Expansion in groups of Lie type", based on the blog cited in my book. Another highly recommended book on expanders is [Kow12].

If you teach expanders to undergraduates, the text [KS11] could be helpful.

14 Page 137, line 2 from above

Change e_- to e^-.

15 Page 147, line 3 from below

Brackets are not needed in the left-hand side

16 Page 153, line 5 from above

$G_i^{t_0}$ should be $G_1^{t_0}$

17 Page 154, lines 3 and 4 from above

$\mathbb{F}^{r+1}$ should be $\mathbb{F}_q^{r+1}$

18 Page 154, line from below

$\mathbb{F}$ should be replaced by $\mathbb{F}_q$ (twice)

19 Sections 5.8 and 5.17: Probabilistic expanders

Puder [Pud15] is a very important contribution to probabilistic expanders.

20 To Section 6.5

The lecture notes of Pisier which were cited as [383] now appeared in a book form [Pis16]. This book contains a lot of material related to different parts of my book, in particular it contains a detailed (and including necessary background) presentation of results on the Pisier-Xu space.

21 To Chapter 7

See Section 30 below.

22 Exercises to Chapter 7

Consider a sequence $\{G_n\}$ of expanders. Let $V = V_n$ be a vertex set of one of them. We partition V into finitely many pieces $P_1^n, \dots, P_k^n$, $k = k(n)$, in such a way that

$$\lim_{n\to\infty} \frac{\max_{1\le i\le k(n)} |P_i^n|}{|V_n|} = 0. \tag{1}$$

We say that such partitions are *without dominating clusters.*

We build a new graph Q_n with the vertex set $\{1, \dots, k(n)\}$ as follows: two vertices are joined by an edge if the corresponding parts are joined by an edge. So Q_n is a minor of G_n if the sets $P_1^n, \dots, P_k^n$ are connected. Observe that the natural quotient map $V_n \to Q_n$ is 1-Lipschitz. So the result is a weak expander if it has bounded geometry.

We get such weak expander if the partitions $P_1^n, \dots, P_k^n$, $k = k(n)$, $n \in \mathbb{N}$ are *uniformly bounded* in the sense that

$$\sup_{n\in\mathbb{N},\ 1\le i\le k(n)} |P_i^n| < \infty. \tag{2}$$

Exercise 22.1. *Can we get a bounded geometry quotient graph in cases where the partitions are not uniformly bounded?*

Exercise 22.2. *Can it happen that the partitions of $\{G_n\}$ are*

- *Not uniformly bounded.*
- *Without dominating clusters.*
- *The corresponding quotients $\{Q_n\}$ have uniformly bounded geometry, but do not form a family of expanders?*

23 Hints to exercises in Chapter 7

To Exercise 22.1. Let $\{F_n\}$ and $\{H_n\}$ be two families of expanders and $G_n = F_n \times H_n$ be their graph-theoretical *Cartesian products*. (This means that $V(G_n) = V(F_n) \times V(H_n)$, and two vertices (u_1, w_1) and (u_2, w_2) are adjacent if and only if either $u_1 = u_2$ and w_1 and w_2 are adjacent in H_n or $w_1 = w_2$ and u_1 and u_2 are adjacent in F_n.) Show that the Cartesian product of two families of expanders is a family of expanders.

To Exercise 22.2. Consider two families of expanders, $\{F_n\}$ and $\{H_n\}$. Attach to each F_n a path length $\ln|V(H_n)|$. Build the family of expanders starting with Cartesian products of F_n and H_n, and such that the 'fibers' corresponding to the path 'decrease' to one vertex as we move from F_n along the path.

24 The notion of "base point"

Base points are mentioned on page 308 and in the Subject Index, but never defined. The *base point* means the same as the *distinguished point* mentioned on page 309.

25 To Section 9.5.1: Superreflexivity and binary trees

Kloeckner [Klo14] found a simple proof of the "if" part of Bourgain's theorem (stated as Theorem 9.43 in the book).

Ostrovskii [Ost14c] proved that (finite) binary trees do not admit uniformly bilipschitz embeddings into diamonds (the converse is also true and is easy to see). This result shows that the Johnson-Schechtman characterization of superreflexivity in terms of diamond is (in a sense) independent from the Bourgain characterization of superreflexivity in terms of binary trees.

Leung, Nelson, Ostrovska, Ostrovskii [LNOO17+] found precise-up-to a logarithmic factor estimates for distortion of embeddings of binary trees into diamonds.

Day [Day41] used a "self-improvement" argument before James [Jam64a].

26 To Section 9.5.2: Further results on test-spaces

It was shown [Ost14c] that any nontrivial word hyperbolic group is a test-space for superreflexivity. In the same paper one can find an easy way of getting one-test-space characterization from a test-space characterization using many finite metric spaces.

27 To Section 9.5.4: Non-local properties

Further results on the metric characterization of the Radon-Nikodým property were obtained in [Ost14a] and [Ost14b].

Important progress in metric characterization of reflexivity was achieved in [MS17+].

An important metric characterization was obtained in [BKL10] prior to publication of my book. A related recent publication is [BCDK+].

28 To Chapter 10

The Lipschitz free spaces are also studied in terms of *Wasserstein* 1 *norm*, see, for example, and important application of such spaces in [NR17, Section 3].

29 To Section 10.4, p. 325

The end of the argument (preceding (10.17)) is a bit sloppy. We need to write something like (normalizing the measures to probability ones and assuming $a_i \geq 0$)

$$\max_{\Theta} \sum a_i |F_i(\Theta)| \geq \sum a_i \int |F_i(\Theta)| d\Theta \geq \sum a_i \left| \int F_i(\Theta) d\Theta \right|.$$

30 To Problem 11.9: The main problem on obstacles for coarse embeddability of bounded geometry metric spaces into ℓ_2

This problem was solved in the negative by Arzhantseva and Tessera [AT15]. They provide two different construction and discuss many interesting related questions.

31 To Problem 11.17: Coarse embeddability of ℓ_2

Problem 11.7 was answered in the negative by F. Baudier, G. Lancien, and T. Schlumprecht in [BLS17+]. The main counterexample is the space constructed by Tsirelson [Tsi74]. (This was one of the spaces which I suggested to look at in my book after posing Problem 11.7.)

My other suggestion was to look at nonreflexive spaces with nontrivial type. This suggestion is questionable as it is known [BS75, BS76] that some of such spaces contain unconditional bases and hence admit coarse embeddings of ℓ_2 into them by the mentioned result of [Ost09].

32 Acknowledgement

I would like to thank Miroslav Bacak for sending me a list of corrections.

33 References

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