Embeddability of locally finite metric spaces into Banach spaces is finitely determined

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A metric space is called *locally finite* if each ball of finite radius in it has finitely many elements. (Finitely generated groups with their word metrics are locally finite metric spaces.)

The main goal of the talk is to describe the tools needed to prove the following results and to mention some of their applications.

Main Theorem: (1) *Let A be a locally finite metric space whose finite subsets admit uniformly bilipschitz embeddings into a Banach space X. Then A admits a bilipschitz embedding into X.*

(2) *Let A be a locally finite metric space whose finite subsets admit uniformly coarse embeddings into a Banach space X. Then A admits a coarse embedding into X.*

Let me recall the definitions.

Let $C < \infty$. A map $f : (A, d_A) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

$$\forall u, v \in A \quad d_Y(f(u), f(v)) \leq C d_A(u, v).$$

A map f is called *Lipschitz* if it is C-Lipschitz for some $C < \infty$.

For a Lipschitz map f we define its *Lipschitz constant* by

$$\text{Lip} f := \sup_{d_A(u,v) \neq 0} \frac{d_Y(f(u), f(v))}{d_A(u, v)}.$$

A map $f : A \to Y$ is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in A \quad r d_A(u, v) \leq d_Y(f(u), f(v)) \leq r C d_A(u, v). \tag{1}$$

A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $C < \infty$. The smallest constant C for which there exist $r > 0$ such that (1) is satisfied is called the *distortion* of f. A set of embeddings is called *uniformly bilipschitz* if they have uniformly bounded distortions.

A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ (observe that this condition implies that ρ_2 has finite values) such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and

$$\forall u, v \in X \ \rho_1(d_X(u, v)) \le d_Y(f(u), f(v)) \le \rho_2(d_X(u, v)). \tag{2}$$

A sequence of embeddings is called *uniformly coarse* if all of them satisfy (2) with the same ρ_1 and ρ_2.

The proof of the Main Theorem is such that it allows to prove similar results for other types of embeddings. For example, it can be used to answer in the negative the following question of Naor and Peres (2011):

Question (Question 10.7 in Naor-Peres (2011)) Let $p \in [1, \infty)$, $p \ne 2$. Does there exist a finitely generated group G for which $\alpha^*_{L_p}(G) \ne \alpha^*_{\ell_p}(G)$?

In this question we use the following definition, and $L_p = L_p(0, 1)$.

Definition: Given a target metric space (X, d_X) the compression exponent of a group G (endowed with its word metric) in X, denoted $\alpha^*_X(G)$, is the supremum over $\alpha \in [0,1]$ for which there exists a Lipschitz function $f : G \to X$ satisfying $d_X(f(x), f(y)) \geq c d_G(x,y)^\alpha$.

Another application of the Main Theorem which I found (2014) is the following:

Any word hyperbolic group with its word metric admits a bilipschitz embedding into any non-superreflexive Banach space (in particular, into any nonreflexive Banach space).

A Banach space $(X, ||\cdot||)$ is called *nonsuperreflexive* if it does not admit a uniformly convex norm $|||\cdot|||$ satisfying the condition

$$\forall x \in X \quad c_1||x|| \leq |||x||| \leq c_2||x||$$

for some $0 < c_1 \leq c_2 < \infty$.

A norm is called *uniformly convex* if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that $|||x||| = |||y||| = 1$ and $|||x-y||| \geq \varepsilon$ imply

$$|||\frac{x+y}{2}||| \leq 1 - \delta.$$

I think that further applications of the Main Theorem have to wait till people will become interested in embeddings into small Banach spaces (like ℓ_p, $p \neq 2, \infty$) or into exotic Banach spaces.

The proof of the Main Theorem starts with the well-known observations belonging to math folklore. (Details and necessary background can be found in my book "Metric Embeddings", Chapter 2). The observation can be described as: embeddability of finite pieces of a metric space A into a Banach space X imply the embeddability of A into a larger Banach space, obtained as some kind of limit related to the Banach space X.

It is convenient to use the following notions:

Let I be an infinite set. A *filter* on I is a subset $\mathcal{F}$ of $\mathcal{P}(I)$ (where $\mathcal{P}(I)$ is the set of all subsets of I) satisfying the following conditions:
(a) $\emptyset \notin \mathcal{F}$. (b) If $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.
(c) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Useful Example: $I = \mathbb{N}$, $\mathcal{F}$ is the set of all subsets of $\mathbb{N}$ having finite complement.

Let Z be a topological space, $f : I \to Z$ be a function. We say that f *converges to* $z \in Z$ *through* $\mathcal{F}$ and write $\lim_{\mathcal{F}} f(x) = z$, if $f^{-1}(U) \in \mathcal{F}$ for every open set U containing z.

An *ultrafilter* $\mathcal{U}$ (on I) is a maximal filter (on I) with respect to inclusion, that is, a filter which is not properly contained in any larger filter.

Lemma: Every filter is contained in an ultrafilter.

An ultrafilter is called *free* if the intersection of all sets of the ultrafilter is empty. (Some authors use *nonprincipal* or *nontrivial* instead of 'free'.)

We can find a free ultrafilter by applying the lemma to the filter of all sets with finite complements in $\mathbb{N}$.

Lemma: Let $\mathcal{U}$ be an ultrafilter on I, K be a compact set, and $f : I \to K$ be a function, then f converges to some point $k \in K$ through $\mathcal{U}$.

This lemma explains the usefulness of the notion of ultrafilter: In many constructions we need to pass to subsequences repeatedly and then consider the diagonal subsequence. Ultrafilters provide what can be called *universal diagonal subsequence*.

Given a family $(X_i)_{i\in I}$ of Banach spaces, the ℓ_∞ *direct sum* of $(X_i)_{i\in I}$ is defined as the space of all bounded collections $(x_i)_{i\in I}$, $x_i \in X_i$ with the vector operations $(x_i)_{i\in I} + (y_i)_{i\in I} = (x_i + y_i)_{i\in I}$, $\alpha(x_i)_{i\in I} = (\alpha x_i)_{i\in I}$, and the norm given by

$$||(x_i)_{i\in I}||_\infty = \sup_{i\in I} ||x_i||_{X_i}.$$

The ℓ_∞ direct sum is denoted by $(\oplus_{i\in I} X_i)_\infty$. It is easy to check that $(\oplus_{i\in I} X_i)_\infty$ is a Banach space.

Let $\mathcal{U}$ be a free ultrafilter on I. By the last lemma the limit $\lim_{\mathcal{U}} ||x_i||_{X_i}$ exists for each $(x_i)_{i\in I} \in (\oplus_{i\in I} X_i)_\infty$. It is easy to see that $\lim_{\mathcal{U}} ||x_i||_{X_i}$ is a seminorm on $(\oplus_{i\in I} X_i)_\infty$. (Recall that a *semi-norm* is like norm except that $||x|| = 0 \Rightarrow x = 0$ is not required.) Let $N_{\mathcal{U}}$ be the subspace of $(\oplus_{i\in I} X_i)_\infty$ on which this seminorm is equal to 0.

One can easily check that $\lim_{\mathcal{U}} ||x_i||_{X_i}$ induces a norm on the quotient space $(\oplus_{i\in I} X_i)_\infty / N_{\mathcal{U}}$. The obtained Banach space is called the *ultra-product* of $(X_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$. We denote it by $(\prod_{i\in I} X_i)_{\mathcal{U}}$ or $(\prod X_i)_{\mathcal{U}}$. If all X_i are the same, the corresponding ultraproduct is also called an *ultrapower* and is denoted $X^{\mathcal{U}}$.

The folklore result which I mentioned is the following:

Proposition: Let A be a metric space which is represented as a union of metric subspaces $\{A_i\}_{i=1}^{\infty}$ satisfying $A_1 \subset A_2 \subset A_3 \subset \dots$. Suppose that $\{A_i\}_{i=1}^{\infty}$ admit uniformly bilipschitz (uniformly coarse) embeddings $f_i : A_i \to X_i$ into Banach spaces $\{X_i\}_{i=1}^{\infty}$. Then A admits a bilipschitz (coarse) embedding into $(\prod X_i)_{\mathcal{U}}$ for any free ultrafilter $\mathcal{U}$. If all X_i are the same, we get an embedding into an ultrapower of X.

The proof is a straightforward application of the definitions (see Proposition 2.21 in my book).

This proposition implies the Main Theorem in some important cases. The most important case is based on the following result:

Theorem: Each separable subspace of any ultrapower of $L_p(0,1)$ is isometric to a subspace of $L_p(0,1)$.

I do not know who proved this theorem first, it has its roots in the paper of Dacuhna-Castelle and Krivine (1972), a complete proof is given in a partially-survey paper of Heinrich (1980).

So for Banach spaces X satisfying the condition: each separable subspace of an ultrapower of X is isometric (actually bilipschitz embeddability suffices) to a subspace of any ultrapower of X the Main Theorem was a folklore result. It is a new result for spaces which do not satisfy the condition. Examples of spaces which do not satisfy this condition are ℓ_p, $p \neq 2, \infty$ and numerous other spaces constructed in Banach spaces as examples/counterexamples to various statements/conjectures.

Since the bilipschitz embeddability is the strongest form of embeddability which we are going to consider, all versions of the Main Theorem (both the stated ones and other versions which one can state, related to Hölder maps, compression exponents, etc) follow from the following result:

Lemma: Let M be a locally finite subset of an ultrapower of a Banach space X. Then there exists a bilipschitz embedding of M into X.