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Weak* closures and derived sets for convex sets in dual Banach spaces

by

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Abstract. The paper is devoted to the convex-set counterpart of the theory of weak* derived sets initiated by Banach and Mazurkiewicz for subspaces. The main result is the following: For every nonreflexive Banach space $\mathcal{X}$ and every countable successor ordinal α, there exists a convex subset A in $\mathcal{X}^*$ such that α is the least ordinal for which the weak* derived set of order α coincides with the weak* closure of A. This result extends the previously known results on weak* derived sets by Ostrovskii (2011) and Silber (2021).

1. Introduction. Let $\mathcal{X}$ be a Banach space. For a subset A of the dual Banach space $\mathcal{X}^*$, we denote the weak* closure of A by $\overline{A}^*$. The *weak* derived set* of A is defined as

$$A^{(1)} = \bigcup_{n=1}^{\infty} \overline{A \cap nB_{\mathcal{X}^*}}^*,$$

where $B_{\mathcal{X}^*}$ is the unit ball of $\mathcal{X}^*$. That is, $A^{(1)}$ is the set of all limits of weak* convergent bounded nets in A. If $\mathcal{X}$ is separable, $A^{(1)}$ coincides with the set of all limits of weak* convergent sequences from A, called the *weak* sequential closure*. The strong closure of a set A in a Banach space is denoted by $\overline{A}$. We set $A^{(0)} := A$.

It was noticed in the early days of Banach space theory by Mazurkiewicz [23] that $A^{(1)}$ does not have to coincide with $\overline{A}^*$ even for a subspace A, and $(A^{(1)})^{(1)}$ can be different from $A^{(1)}$. In this connection, it is natural to introduce derived sets for all ordinals: (1) if $A^{(\alpha)}$ has already been defined, then $A^{(\alpha+1)} := (A^{(\alpha)})^{(1)}$; (2) if α is a limit ordinal and $A^{(\beta)}$ has already been defined for all $\beta < \alpha$, then

$$A^{(\alpha)} := \bigcup_{\beta<\alpha} A^{(\beta)}. \tag{1.1}$$

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The study of weak* derived sets was initiated by Banach and Mazurkiewicz (see [23, 3]). Its early developments are discussed at length in the Appendix to the classical monograph by Banach [3]. Later, this study was continued by many authors and found significant applications. Since the well-known survey [34] of the fields of Banach space theory initiated by [3] does not mention developments stemming from Banach's "Appendix" [3, Annexe], it seems beneficial to present here a short historical account.

Banach and Mazurkiewicz were primarily interested in the case of a separable Banach space $\mathcal{X}$. Banach asked whether the weak* sequential closure of a subspace may not be weak* sequentially closed, and Mazurkiewicz [23] gave an affirmative answer to this question. This result was the reason for Banach to introduce weak* sequential closures of all transfinite orders.

In [3, Annexe] Banach proved that weak* sequential closures of finite orders do not have to be weak* sequentially closed. Furthermore, Banach stated that in his paper, which was going to appear in *Studia Math.*, volume 4, he proved a similar result for $\mathcal{X} = c_0$ and an arbitrary countable ordinal. However, this paper has never been published. A possible explanation can be that Banach found a mistake in his proof when it was too late to delete the statement and the reference in [3]. It is regrettable that the story was left uncommented in the reprint of [3] in [4] and the survey [34] because the editors of [4] might have known the actual story.

In the late 20s and early 30s, Banach and his school focused on the sequential approach to weak* topology and did not use the notion of weak* topology. The subject of what is now called general topology already existed [2], but was not yet well known. Using general topology a significant part of the theory was made more elegant (see an account in [10]). However, the sequential approach developed in [3, Annexe] has its advantages and has led to significant applications. An early application of weak* sequential closures to the study of sets of uniqueness for Fourier series was discovered by Piatetski-Shapiro [36], and further developed in [19] and [22].

As for further development of theoretical aspects of weak* sequential closures, it is worth pointing out that Banach's claim mentioned above (see [3, Annexe, §1]) was proved in 1968 by McGehee [24], using results by Piatetski-Shapiro [36]. At the same time, Sarason [42, 43] proved similar results for some other spaces.

Davis and Johnson [6] developed an essential tool for investigation of non-quasi-reflexive Banach spaces (that is, spaces for which the canonical image of $\mathcal{X}$ in $\mathcal{X}^{**}$ has infinite codimension). This tool was used by Godun [13] to prove that for any finite ordinal the dual of any non-quasi-reflexive Banach space contains a subspace whose weak* sequential closures of finite orders form a strictly increasing sequence (this result was rediscovered later by

Moscatelli [26]). Godun [12] also made an attempt to prove similar results for infinite countable ordinals, but his argument contains gaps.

A result which completes the investigation for separable Banach spaces was proved in [29] for general non-quasi-reflexive separable Banach spaces: it was shown that, for every separable non-quasi-reflexive Banach space $\mathcal{X}$ and every countable ordinal κ, the space $\mathcal{X}^*$ contains a linear subspace Γ for which $\Gamma^{(\kappa)} \neq \Gamma^{(\kappa+1)} = \mathcal{X}^*$. This result completes the investigation for separable Banach spaces for the following reasons: (1) It is easy to see that if $\mathcal{X}$ is a separable quasi-reflexive Banach space and Γ is a subspace of $\mathcal{X}^*$, then $\Gamma^{(1)} = \overline{\Gamma}^*$. (2) It is known (a proof is sketched in [3, p. 213]) that for a separable Banach space $\mathcal{X}$ and any convex subset $A \subset \mathcal{X}^*$, there is a countable ordinal α such that $A^{(\alpha)} = \overline{A}^*$.

This result of [29] is presented in [14]. Regrettably, the historical information on weak* sequential closures in [14] is inaccurate.

Meanwhile, the theory of weak* sequential closures found applications in many different fields:

- the structure theory of Fréchet spaces [5, 8, 25, 27, 30];
- the Borel and Baire classification of linear operators, including applications to the theory of ill-posed problems [40, 35, 38, 39];
- operator theory [44];
- the theory of universal Markushevich bases [37, 14].

The survey [31] contains a more detailed historical account on weak* sequential closures, which was up-to-date in 2000.

It is worth mentioning that the proof of nonexistence of universal Markushevich bases in [37] uses the existence of subspaces satisfying $\Gamma^{(\kappa)} \neq \Gamma^{(\kappa+1)} = \mathcal{X}^*$ in the same way as Szlenk [47] uses the existence of reflexive spaces with an arbitrarily large Szlenk index in his proof of nonexistence of universal reflexive Banach spaces.

Recently, sequential closures and derived sets became objects of interest in some other areas, such as:

- extension problems for holomorphic functions on dual Banach spaces [11];
- valuations [1];
- mathematical economics [7];
- the duality operators/spaces [41].

Several years after the work of Banach and Mazurkiewicz, Krein and Šmulian [20] started to develop a similar theory for convex sets. Since subspaces are convex sets, and so there is no need to generalize examples of subspaces with long chains of increasing weak* sequential closures, Krein and Šmulian focused on results characterizing weak* closed convex sets.

The fact that the existing theory of Krein–Šmulian together with the examples listed above for subspaces does not contain answers to all questions which are natural to ask about convex sets was noticed by García–Kalenda–Maestre [11] in their study of extension problems for holomorphic functions on dual Banach spaces. They initiated a further development of the theory for convex sets by asking the following question [11, Question 6.3]: Let $\mathcal{X}$ be a quasi-reflexive Banach space. Is $A^{(1)}$ equal to $\overline{A}^*$ for each (absolutely) convex set $A \subset \mathcal{X}$? This question was answered by Ostrovskii [32], who proved that (1) the answer is "yes" for absolutely convex sets, (2) for an arbitrary nonreflexive Banach space $\mathcal{X}$ (quasi-reflexive or not), there is a convex set $A \subset \mathcal{X}$ for which $A^{(1)} \neq A^{(2)}$. This result showed that the theory for convex sets is different from the theory for subspaces: for convex sets quasi-reflexivity does not imply that $A^{(1)} = \overline{A}^*$. Silber [45] developed this result further by proving that, for an arbitrary nonreflexive space $\mathcal{X}$ and each $n \in \mathbb{N}$, the space $\mathcal{X}^*$ contains a convex subset A for which n is the least ordinal satisfying $A^{(n)} = \overline{A}^*$ and a convex subset D for which $\omega_0 + 1$ is the least ordinal satisfying $D^{(\alpha)} = \overline{D}^*$, where ω_0 is the least infinite ordinal.

Our goal is to develop this theory further by proving the following result.

THEOREM 1.1. *Let $\mathcal{X}$ be a nonreflexive Banach space and κ be a countable ordinal. Then there exists a convex subset $A \subset \mathcal{X}^*$ such that $A^{(\kappa)} \neq A^{(\kappa+1)} = \overline{A}^*$.*

This theorem is proved in Section 5.

The following question asked in [45] remains unanswered.

QUESTION 1.2 ([45, Section 3, Question 1]). *Does there exist a convex subset A in the dual to a separable Banach space for which the least ordinal α satisfying $A^{(\alpha)} = \overline{A}^*$ is a limit ordinal?*

It should be mentioned that it is an easy consequence of the Baire theorem that this cannot happen if we additionally require that A is a subspace. To the best of my knowledge, Godun [12] was the first to make this observation (later it was repeatedly rediscovered, see [19, 15]).

2. Trees. Our construction of sets is quite different from the one in [32, 45]. For our construction of sets whose existence is stated in Theorem 1.1, we shall use the notion of a tree defined as in set theory; see [16, p. 114] and [21, p. 201]. Namely, a *tree* $(F, <)$ is a partially ordered set such that for each $u \in F$ the set $\{w : w < u\}$ is well-ordered (each of its subsets has a least element). We need only trees satisfying a much stronger condition: each set $\{w : w < u\}$ is a finite linearly ordered set. Let F be such a tree. We introduce on it a graph structure as follows: Two vertices $u, v \in F$ are *adjacent* if and only if they are comparable and the set of vertices between them is empty. For a vertex $u \in F$, its *up-degree* is defined as the cardinality of the set of vertices v adjacent to u and satisfying $u < v$. Each such v is called an *up-neighbor* of u, written $v \succ u$. Likewise, the *down-degree* of a vertex $u \in F$ is defined as the cardinality of the set of vertices v adjacent to u and satisfying $v < u$. Each such v is called a *down-neighbor* of u, and we write $v \prec u$. It is clear that the down-degree of a vertex in F can be either 0 or 1, while the up-degree can be any cardinal. Vertices with down-degree 0 are called *initial*. Vertices with up-degree 0 are called *terminal*.

Note that our definition is in conflict with graph-theoretic terminology, where trees are required to be connected, and disconnected trees are called *forests*. Nevertheless, in this paper we shall use the term *tree*.

It has to be pointed out that although our terminology and constructions are related to the ones in [9, p. 161], [17, Section 2], [18], [19], [28, Section 2], and [48], they are different.

Throughout the paper, all ordinals (except the least uncountable ordinal ω_1) are assumed to be countable. If $\{K_i\}_{i=1}^k$ are partially ordered sets, we denote by $\biguplus_{i=1}^k K_i$, where $k \in \mathbb{N}$ of $k = \infty$, the partially ordered set defined as follows. As a set, it is a disjoint union of $\{K_i\}_{i=1}^k$, with the partial order which on each K_i coincides with the original partial order, and elements of different K_i are incomparable. We shall also use the self-explanatory notation

$$\left(\biguplus_{i=1}^{k} K_i\right) \uplus \left(\biguplus_{i=k+1}^{\infty} K_i\right).$$

An element w of a tree F is called a *root* of F if $w \le u$ for every $u \in F$.

To achieve our goal, we introduce a family of trees labelled by countable ordinals α, including all finite ordinals. For each ordinal $\alpha < \omega_1$, we construct a tree F_α of the type described above. The tree F_α will be called a *tree of order* α. Our construction of F_α is inductive.

The tree F_0 of order 0 is a one-element partially ordered set. So it contains one vertex and no edges.

Suppose that we have already defined the tree F_α of order α. When α is a successor ordinal or $\alpha = 0$, the tree $F_{\alpha+1}$ of order $\alpha + 1$ is defined in two steps: first we consider $\biguplus_{i=1}^\infty K_i$, where each K_i is a copy of F_α. In the second step, we add to this union a root, that is, an element smaller than all elements of $\biguplus_{i=1}^\infty K_i$. Observe that the root of $F_{\alpha+1}$ is adjacent to initial vertices of each copy of F_α, and has no other adjacent vertices.

If $\beta < \omega_1$ is a limit ordinal, we consider a strictly increasing sequence $\{\beta_n\}_{n=1}^\infty$ of successor ordinals converging to β and define F_β as $\biguplus_{n=1}^\infty F_{\beta_n}$. Thus F_β is disconnected, with $\aleph_0$ (the least infinite cardinal) connected components.

Finally, to define $F_{\alpha+1}$ when α is a limit ordinal, we add to the graph F_α a root, that is, one more vertex smaller than all vertices of F_α. The definition of F_α implies that this new vertex has up-degree $\aleph_0$.

Observe that the up-degrees of vertices in the trees F_α can only be 0 or $\aleph_0$. This follows by observing that each new vertex (that is, not a copy of one introduced before) which is used in the construction of F_α for $\alpha \geq 1$ has up-degree $\aleph_0$. Only vertices obtained from the vertex forming F_0 by repeated copying have up-degree 0.

Note that F_n for n finite is what is called the (infinitely) *countably branching tree of depth* n.

For a tree F of the type described above, we define the *derived tree* F^1 as the subgraph of F obtained by deleting from F all infinite sets of terminal vertices having the same down-neighbor. To avoid confusion, we repeat the same definition in more detail: we remove a terminal vertex v if it has a down-neighbor vertex w and w has infinitely many up-neighbors that are terminal vertices.

Having defined the derived tree F^β of order β, we define the derived tree of order $\beta+1$ as $(F^\beta)^1$. If β is a limit ordinal and we have already defined derived trees for smaller ordinals, we set $F^\beta = \bigcap_{\gamma<\beta} F^\gamma$.

Note that the derived tree $(F_\alpha)^\gamma$ (we write F_α^γ since it does not create any confusion) of order γ is not necessarily of the form F_β for any ordinal β. For example, $F_{\omega_0}^{\omega_0}$ (where ω_0 is the least infinite ordinal) is a collection of isolated vertices of cardinality $\aleph_0$.

The following two lemmas contain results on derived trees needed for our construction.

Lemma 2.1.

(1) *Let n and k, $1 \leq k \leq n$, be finite ordinals. Then $(F_n)^k = F_{n-k}$.*
(2) *Let α be any ordinal. Then $(F_{\alpha+1})^\alpha = F_1$ and $(F_{\alpha+1})^{\alpha+1} = F_0$.*
(3) *Let α be a limit ordinal and $\{\beta_n\}_{n=1}^\infty$ be an increasing sequence of successor ordinals converging to α, used to define F_α. Then, for $\beta < \alpha$,*

$$(F_\alpha)^\beta = \Big(\biguplus_{n,\,\beta\geq\beta_n} F_0\Big) \uplus \Big(\biguplus_{n,\,\beta<\beta_n} (F_{\beta_n})^\beta\Big), \tag{2.1}$$

and

$$(F_\alpha)^\alpha = \biguplus_{i=1}^{\infty} F_0.$$

Proof. (1) The case of finite ordinals follows easily from the definition of F_n.

It is also easy to see from the definition of F_α for a successor ordinal that the conditions $(F_{\alpha+1})^\alpha = F_1$ and $(F_{\alpha+1})^{\alpha+1} = F_0$ imply $(F_{\alpha+2})^{\alpha+1} = F_1$ and $(F_{\alpha+2})^{\alpha+2} = F_0$.

Before proving (2) for limit ordinals (assuming we have proved it for all smaller ordinals), we need to prove the equality (2.1).

To get (2.1) we use the obvious equality $F_0^{\gamma} = F_0$ for every γ, its generalization

$$\Big(\biguplus_{n,\ \beta\geq\beta_n} F_0\Big)^{\gamma} = \Big(\biguplus_{n,\ \beta\geq\beta_n} F_0\Big),$$

and the definition of F_α. The equality (2.1) implies that $F_{\alpha+1}^{\alpha} = F_1$ and $F_{\alpha+1}^{\alpha+1} = F_0$ for a limit ordinal α, provided statement (2) is known for all ordinals $\beta < \alpha$.

This implies all formulas stated in Lemma 2.1. ∎

Lemma 2.2. *If a vertex in $(F_\alpha)^\beta$ has infinitely many up-neighbors which are terminal vertices of $(F_\alpha)^\beta$, then all of its up-neighbors in F_α are terminal vertices of $(F_\alpha)^\beta$.*

Proof. We use induction on α. For finite α, the statement immediately follows from the description of F_α^β in Lemma 2.1(1).

If the statement is true for all ordinals which are strictly less than a limit ordinal α, then it is true for F_α. In fact, since F_α consists of components which are F_β for $\beta < \alpha$, and derived trees of disconnected trees are taken componentwise, the conclusion follows.

Now we assume that the statement is true for F_α, and derive it for $F_{\alpha+1}$. There are two cases:

(a) α is a limit ordinal,
(b) α is a successor ordinal,

In both cases the induction hypothesis implies that the condition holds for all vertices of all derived trees except possibly the initial vertex.

Case (a): For the initial vertex the condition is also satisfied because Lemma 2.1(3) implies that for $\beta < \alpha$, the initial vertex has finitely many terminal up-neighbors in the derived tree $F_{\alpha+1}^\beta$. On the other hand, all up-neighbors of the initial vertex are terminal in the derived tree $(F_{\alpha+1})^\alpha$.

Case (b): In this case, each of the derived trees consists of $\aleph_0$ copies of the same tree attached to the initial vertex, and the conclusion follows. ∎

3. Reduction to the separable case. The main part of the proof of Theorem 1.1 is its proof for a separable Banach space with a basis satisfying special conditions. In this section we show that this special case implies the general case of Theorem 1.1.

We need the following notation. Let $\mathcal{Z}$ be a closed subspace in a Banach space $\mathcal{X}$ and $E : \mathcal{Z} \to \mathcal{X}$ be the natural isometric embedding. Then $E^* : \mathcal{X}^* \to \mathcal{Z}^*$ is a quotient mapping which maps each functional in $\mathcal{X}^*$ to its restriction to $\mathcal{Z}$. Let A be a subset of $\mathcal{Z}^*$. It is clear that $D = (E^*)^{-1}(A)$ is the set of all extensions over $\mathcal{X}$ of functionals in A.

LEMMA 3.1 (cf. [29, 32]). *For any ordinal α we have*

$$D^{(\alpha)} = (E^*)^{-1}(A^{(\alpha)}), \tag{3.1}$$

where the derived set $D^{(\alpha)}$ is taken in $\mathcal{X}^*$, and the derived set $A^{(\alpha)}$ in $\mathcal{Z}^*$.

Proof. The inclusion $D^{(\alpha)} \subset (E^*)^{-1}(A^{(\alpha)})$ follows from the weak* continuity of the operator E^* using transfinite induction.

To prove the inverse inclusion by transfinite induction, it suffices to show that for every bounded net $\{f_\nu\} \subset \mathcal{Z}^*$ with $w^*\text{-}\lim_\nu f_\nu = f$ and every g in $(E^*)^{-1}(\{f\})$ there exist $g_\nu \in (E^*)^{-1}(\{f_\nu\})$ such that some subnet of $\{g_\nu\}$ is bounded and weak* convergent to g. Let h_ν be such that $h_\nu \in (E^*)^{-1}(\{f_\nu\})$ and $\|h_\nu\| = \|f_\nu\|$ (Hahn–Banach extensions). Then $\{h_\nu\}_\nu$ is a bounded net in $\mathcal{X}^*$. Hence it has a weak* convergent subnet; let h be its limit. Then $g - h \in (E^*)^{-1}(\{0\})$, and therefore $g_\nu = h_\nu + g - h$ is as desired. ■

Reduction of Theorem 1.1 to the special case. We are going to use the following result [33, 46]: If a Banach space $\mathcal{X}$ is nonreflexive, then it contains a bounded basic sequence $\{z_i\}_{i=1}^\infty$ such that $\|z_i\| \ge 1$ for every $i \in \mathbb{N}$, but

$$\sup_{1\le k<\infty} \Big\| \sum_{i=1}^k z_i \Big\| = C < \infty. \tag{3.2}$$

Let $\mathcal{Z}$ be the closed linear span of $\{z_i\}_{i=1}^\infty$. We show that Lemma 3.1 implies that to prove Theorem 1.1 it suffices to find a convex subset $A \subset \mathcal{Z}^*$ such that $\overline{A}^* = A^{(\kappa+1)} \ne A^{(\kappa)}$. In fact, if we construct such an A, we let $D = (E^*)^{-1}(A)$. By Lemma 3.1 we have, $D^{(\kappa)} = (E^*)^{-1}(A^{(\kappa)})$ and $D^{(\kappa+1)} = (E^*)^{-1}(A^{(\kappa+1)})$. Since each functional has a continuous extension, $A^{(\kappa+1)} \ne A^{(\kappa)}$ implies $D^{(\kappa+1)} \ne D^{(\kappa)}$.

To show that $\overline{A}^* = A^{(\kappa+1)}$ implies $\overline{D}^* = D^{(\kappa+1)}$ we observe that $\overline{A}^* = A^{(\kappa+1)}$ implies that $A^{(\kappa+1)} = A^{(\kappa+2)}$. By Lemma 3.1 the last equality implies $D^{(\kappa+1)} = D^{(\kappa+2)}$. By the Krein–Šmulian theorem [10, p. 429], the condition $D^{(\kappa+1)} = D^{(\kappa+2)}$ implies $\overline{D}^* = D^{(\kappa+1)}$. ■

For this reason from now on our goal is to prove Theorem 1.1 for $\mathcal{X} = \mathcal{Z}$.

Later we shall need some more notation and observations related to the space $\mathcal{Z}$. Let $\{z_i^*\}_{i=1}^\infty \subset \mathcal{Z}^*$ be the biorthogonal functionals of $\{z_i\}_{i=1}^\infty$. Let z^{**} be a weak* cluster point of the sequence $\{\sum_{i=1}^k z_i\}_{k=1}^\infty$ in $\mathcal{Z}^{**}$.

We will need the following observations about these vectors:

(3.3) $|z^{**}(x)| \le C\|x\|$ for every $x \in \mathcal{Z}^*$.

This is an immediate consequence of $\|z^{**}\| \le C$ which follows from (3.2).

(3.4) If x is a linear combination of $\{z_i^*\}_{i=1}^{\infty}$ with nonnegative coefficients, then $z^{**}(x) \geq c\|x\|$ for some $c > 0$.

In fact, let $x = \sum_{i=1}^{k} a_i z_i^*$. Then $z^{**}(x) = \sum_{i=1}^{k} a_i$. On the other hand,

$$\|x\| = \Big\| \sum_{i=1}^{k} a_i z_i^* \Big\| \leq \sup_i \|z_i^*\| \sum_{i=1}^{k} a_i.$$

Since $\{z_i\}$ is a basic sequence satisfying $\|z_i\| \geq 1$, $\sup_i \|z_i^*\|$ is finite, and the conclusion follows.

Note that analysis of the proof of [33, 46] leads to reasonably small absolute bounds for $\sup_i \|z_i^*\|$ and C from above, but we do not need such bounds.

4. Construction of a suitable convex set. Now we construct the set A whose existence is claimed in Theorem 1.1. We fix a countable ordinal κ and let $\alpha = \kappa + 1$.

We introduce an injective map of F_α into $\mathbb{N}$ and identify each vertex of F_α with its image in $\mathbb{N}$. We may and will assume that if $u < v$ in F_α, then the images $\bar{u}$ and $\bar{v}$ of u, v in $\mathbb{N}$ satisfy $\bar{u} < \bar{v}$. In fact, to establish such identification we reserve for connected components of F_α pairwise disjoint infinite subsets in $\mathbb{N}$ (we reserve the whole $\mathbb{N}$ if F_α is connected). Then we assign to the initial vertex of each component of F_α the least number of the corresponding subset and delete the initial vertices from F_α. Unless an initial vertex in a component K of F_α was a terminal vertex, its deletion will split K into countably many (incomparable) components. We reserve for these components pairwise disjoint infinite subsets of the infinite subset reserved for K, and continue in the obvious way.

For each terminal vertex $v \equiv n_k \in \mathbb{N}$ ($\equiv$ means that we identify the vertex v and the number n_k) of F_α we consider the path joining v to the initial vertex of the component of F_α containing v. Let the path be $n_k, \ldots, n_1$. In the path we list vertices only and observe that (by the result of the previous paragraph) $n_k, \ldots, n_1$ is a decreasing sequence in $\mathbb{N}$.

For a terminal vertex v of F_α, corresponding to $\{n_k, \ldots, n_1\}$, we introduce a vector in $\mathcal{Z}^*$ given by

$$z^*(v) = \sum_{i=1}^{k} n_{i-1} z_{n_i}^*, \tag{4.1}$$

where we set $n_0 = n_1$.

Let

$$X = X_\alpha = \{z^*(v) : v \text{ is a terminal vertex in } F_\alpha\}, \tag{4.2}$$

and let $A = A_\alpha = \operatorname{conv}(X)$.

Our next goal is to analyze the structure of the weak* derived sets of A.

In this connection, for each countable ordinal β, we define a *shortening* X^β of the set X as

$$X^\beta = \Big\{ \sum_{n_i \in F_\alpha^\beta} n_{i-1} z^*_{n_i} : \sum_{i=1}^{k} n_{i-1} z^*_{n_i} \in X \Big\},$$

where $n_i \in F_\alpha^\beta$ means that the vertex of F_α corresponding to n_i belongs to the derived tree $(F_\alpha)^\beta$ of order β (defined in Section 2). We let $X^0 = X$.

Remark 4.1. Observe that each vector y in any X^β including X^0 is supported on the vertex set of a finite path in F_α whose vertex set is linearly ordered (recall that the vertex set of F_α is partially ordered). We denote the largest vertex in this path by $v(y)$.

Corollary 4.2 (of Lemmas 2.1 and 2.2). *The set $X^{\beta+1} \setminus \bigcup_{0 \le \gamma \le \beta} X^\gamma$ is nonempty for every $\beta < \alpha$. Furthermore, this difference contains a vector y whose support does not contain the support of any of the vectors in $\bigcup_{0 \le \gamma \le \beta} X^\gamma$.*

Proof. Lemma 2.1 implies that the derived trees $\{F_\alpha^\lambda\}_\lambda$ stabilize only starting from $\lambda = \alpha$. Therefore, for any $\beta < \alpha$, we have $F_\alpha^{\beta+1} \ne F_\alpha^\beta$. This yields the existence in F_α^β of an infinite set of terminal vertices which have the same down-neighbor v. By Lemma 2.2, all up-neighbors of v in F_α are terminal vertices of F_α^β.

Let us consider any vector in $X = X_\alpha$ having v in its support. Such a vector can be easily obtained by extending the path joining v to the initial vertex of the component of F_α containing v to a maximal monotone path in F_α. Let $n_k, \dots, n_1 \in \mathbb{N}$ be the vertex set of this path listed in decreasing order using the identification described above. (The finiteness of the path is immediate from Lemma 2.1.)

The vertex v corresponds to one of the numbers $n_{k-1}, \dots, n_1$. The vector $\sum_{i=1}^k n_{i-1} z^*_{n_i}$ belongs to $X = X_\alpha$, and $y = \sum_{n_i \in F_\alpha^{\beta+1}} n_{i-1} z^*_{n_i}$ is in $X^{\beta+1}$ but not in X^γ for $\gamma \le \beta$.

It remains to show that $\operatorname{supp} y$ does not contain the support of any vector $x \in X^\gamma$, $0 \le \gamma \le \beta$.

Assume the contrary: let x be such that $\operatorname{supp} x \subset \operatorname{supp} y$, and the support of x is at least as large as the support of any $\tilde{x}$ which is contained in X^γ for some $0 \le \gamma \le \beta$ and satisfies $\operatorname{supp} \tilde{x} \subset \operatorname{supp} y$. For such x, let γ be the minimal ordinal for which $x \in X^\gamma$. Clearly, $\gamma > 0$, because no vertex in $\operatorname{supp} y$ is terminal in F_α. Let $\sum_{i=1}^j m_{i-1} z^*_{m_i} \in X$ be such that $x = \sum_{m_i \in F_\alpha^\gamma} m_{i-1} z^*_{m_i} = \sum_{i=1}^l m_{i-1} z^*_{m_i}$. Then $m_{l+1} \notin \operatorname{supp} y$ because of the maximality of $\operatorname{supp} x$. The vertex m_{l+1} belongs to an infinite family of vertices which are terminal in F_α^η for some $\eta < \gamma$, as otherwise m_{l+1} would belong to F_α^γ.

By Lemma 2.2, all up-neighbors of m_l are terminal vertices of F_α^η. We get a contradiction because at least one of them belongs to F_α^β. ■

5. Proof of Theorem 1.1. The main steps in our proof of Theorem 1.1 are the following:

(A) For every $\beta \le \alpha$,

$$\operatorname{conv}\Big(\bigcup_{0\le\gamma\le\beta} X^\gamma\Big) \subset A^{(\beta)}. \tag{5.1}$$

(B) For every $\beta \le \alpha$,

$$A^{(\beta)} \subset \overline{\operatorname{conv}\Big(\bigcup_{0\le\gamma\le\beta} X^\gamma\Big)}. \tag{5.2}$$

(C) If $\beta < \alpha$, then $X^{\beta+1} \setminus \overline{\operatorname{conv}(\bigcup_{0\le\gamma\le\beta} X^\gamma)} \ne \emptyset$.

(D) The weak* sequential closure of $B := \overline{\operatorname{conv}(\bigcup_{0\le\gamma\le\alpha} X^\gamma)}$ coincides with B. Therefore B is weak* closed.

(E) The inclusion in (5.2) becomes an equality if α is a successor ordinal and $\beta = \alpha$.

5.1. Deduction of Theorem 1.1 from (A)–(E). Recall that we use $\alpha = \kappa + 1$. By (E), we get

$$A^{(\kappa+1)} = \overline{\operatorname{conv}\Big(\bigcup_{0\le\gamma\le\kappa+1} X^\gamma\Big)}. \tag{5.3}$$

By (D), $A^{(\kappa+1)}$ is weak* closed and thus coincides with $\overline{A}^*$. On the other hand, combining (A), (B), and (C) we get $A^{(\kappa)} \ne A^{(\kappa+1)}$. □

5.2. Proof of item (A). Since convexity is preserved under weak* sequential closure, to prove (A) by induction it suffices to show that $X^\beta \subset A^{(\beta)}$ for every $\beta \le \alpha$.

The inclusion $X^1 \subset A^{(1)}$ can be derived from the definitions as follows. The definitions imply that $y \in X^1 \setminus X^0$ if and only if there is an infinite sequence $\{v_k\} \subset F_\alpha$ of terminal vertices having the same down-neighbor u, such that $y = z^*(u)$ (for nonterminal vertices we use the same notation as in (4.1)). Let $x_k = z^*(v_k)$. Then, as is easy to see, $x_k \in X$. On the other hand, the definition (4.1) implies that $x_k = z^*(v_k) = z^*(u) + n_{i-1} z_{n_i}^* = y + n_{i-1} z_{n_i}^*$, where n_{i-1} is the same for all k and $\{z_{n_i}^*\}_i$ is some subsequence of $\{z_i^*\}_{i=1}^\infty$. Therefore, $y = w^*\text{-}\lim_{n\to\infty} x_n$.

In a similar way, if we know that $X^\beta \subset A^{(\beta)}$, we derive $X^{\beta+1} \subset A^{(\beta+1)}$ from the fact that each element of $X^{\beta+1} \setminus X^\beta$ is the limit of a weak* convergent sequence of elements of X^β.

The definition $(F_\alpha)^\beta = \bigcap_{\gamma<\beta}(F_\alpha)^\gamma$ for the derived tree of order β with a limit ordinal β implies that

$$X^\beta \subset \bigcup_{\gamma<\beta} X^\gamma \tag{5.4}$$

for a limit ordinal β. Combining (5.4) with the definition of the weak* derived set $A^{(\beta)}$ for a limit ordinal β (see (1.1)), we find that the validity of the inclusion $X^\tau \subset A^{(\tau)}$ for all ordinals $\tau < \beta$ implies its validity for a limit ordinal β.

5.3. Proof of item (B). We prove (5.2) by induction. Of course, we have the inclusion for $\beta = 0$.

The next step is to suppose that

$$A^{(\beta)} \subset \overline{\text{conv}\Big(\bigcup_{0\le\gamma\le\beta} X^\gamma\Big)},$$

and use it to derive

$$A^{(\beta+1)} \subset \overline{\text{conv}\Big(\bigcup_{0\le\gamma\le\beta+1} X^\gamma\Big)}.$$

To achieve this it is clearly enough to show that

$$\Big(\text{conv}\Big(\bigcup_{0\le\gamma\le\beta} X^\gamma\Big)\Big)^{(1)} \subset \overline{\text{conv}\Big(\bigcup_{0\le\gamma\le\beta+1} X^\gamma\Big)}. \tag{5.5}$$

The proof of the step $\beta \to \beta+1$ will complete the proof of (5.2), because for a limit ordinal β the inclusion (5.2) follows immediately from the definition of $A^{(\beta)}$ for a limit ordinal β, provided (5.2) has already been proved for all $\tau < \beta$.

So we prove the step $\beta \to \beta + 1$.

Since $\text{conv}(\bigcup_{0\le\gamma\le\beta} X^\gamma)$ is a subset of the dual of a separable Banach space, any element of its weak* derived set is a weak* limit of a bounded sequence of the form

$$\Big\{\sum_{x\in W} a_{x,i}x\Big\}_{i=1}^\infty, \quad \text{where} \quad a_{x,i} \ge 0, \quad \sum_{x\in W} a_{x,i} = 1, \tag{5.6}$$

where $W = \bigcup_{0\le\gamma\le\beta} X^\gamma$ and the set $\{a_{x,i}\}_{x\in W}$ is finitely nonzero for any $i \in \mathbb{N}$.

For each $x \in W$ we consider the vertex $v(x)$ in F_α (see the definition in Remark 4.1). It can happen that for some $x \in W$ the vertex $v(x)$ is an initial vertex. We denote the set of all such $x \in W$ by I.

For $x \in W \setminus I$ denote by $v(y)$ the down-neighbor of $v(x)$ in F_α and denote by $y = y(x)$ the vector in $\mathcal{Z}^*$ obtained if we replace by 0 the component of x corresponding to $v(x)$, so that $v(y)$ agrees with the definition in Remark 4.1.

We group the summands of $\sum_x a_{x,i}x$ for $x \in W \setminus I$ according to vectors $y = y(x)$ defined in the previous paragraph. We denote the set of all such vectors y obtained for different $x \in W \setminus I$ by P. We can write

$$\sum_{x\in W} a_{x,i}x = \sum_{x\in I} a_{x,i}x + \sum_{y\in P} \sum_{\{x\in W:\, v(x)\succ v(y)\}} a_{x,i}x,$$

where $v(x) \succ v(y)$ means that $v(y)$ is a down-neighbor of $v(x)$.

We may assume that $\lim_{i\to\infty} \sum_{\{x\in W:\, v(x)\succ v(y)\}} a_{x,i}$ exists for every $y \in P$ and denote this limit by s_y. We may also assume that $\lim_{i\to\infty} a_{x,i}$ exists for every $x \in W$ and denote this limit by p_x.

Lemma 5.1. *We have* $\sum_{x\in I} p_x + \sum_{y\in P} s_y = 1$.

Proof. In fact, suppose $\sum_{x\in I} p_x + \sum_{y\in P} s_y = \sigma < 1$. We will show that this contradicts the boundedness of $\{\sum_{x\in W} a_{x,i}x\}_{i=1}^\infty$.

For any finite subsets $G \subset I$ and $H \subset P$, and every $\varepsilon > 0$, there is $j \in \mathbb{N}$ such that

$$\sum_{x\in G}(a_{x,i} - p_x) + \sum_{y\in H}\Big(\Big(\sum_{\{x\in W:\, v(x)\succ v(y)\}} a_{x,i}\Big) - s_y\Big) < \varepsilon \quad \text{for } i \ge j.$$

Therefore

$$\sum_{x\in I\setminus G} a_{x,i} + \sum_{y\in P\setminus H}\Big(\sum_{\{x\in W:\, v(x)\succ v(y)\}} a_{x,i}\Big) > 1 - (\sigma+\varepsilon) \quad \text{for } i \ge j.$$

For any $M \in \mathbb{N}$, we can pick H such that for all $y \in P \setminus H$ the natural number corresponding to $v(y)$ in the identification described in Section 4 is at least M. (For this and the next statement we need to recall that $n_0 = n_1$ in (4.1).)

Similarly, we can pick G in such a way that for all $x \in I \setminus G$, the natural number corresponding to $v(x)$ (recall that $n_0 = n_1$, see the line after (4.1)) is at least M. Then, by (3.3),

$$\begin{aligned}\Big\|\sum_{x\in W} a_{x,i}x\Big\| &\ge \frac{1}{C}\, z^{**}\Big(\sum_{x\in W} a_{x,i}x\Big)\\ &\ge \frac{1}{C}\, z^{**}\Big(\sum_{x\in I\setminus G} a_{x,i}x + \sum_{y\in P\setminus H}\Big(\sum_{\{x\in W:\, v(x)\succ v(y)\}} a_{x,i}x\Big)\Big)\\ &\ge \frac{M(1-(\sigma+\varepsilon))}{C}.\end{aligned}$$

Since this can be done for every $M \in \mathbb{N}$ and every $\varepsilon > 0$, we conclude that $\{\sum_{x\in W} a_{x,i}x\}_{i=1}^\infty$ is unbounded. This contradiction proves the lemma. ∎

LEMMA 5.2. *The vectors* $\{\sum_{\{x\in W:\,v(x)\succ v(y)\}} a_{x,i}\}_{y\in P}$ *converge to* $\{s_y\}_{y\in P}$ *strongly in* $\ell_1(P)$ *and the vectors* $\{a_{x,i}\}_{x\in I}$ *converge to* $\{p_x\}_{x\in I}$ *in* $\ell_1(I)$.

Lemma 5.2 is an immediate consequence of the fact that a sequence $\{v_i\}$ of normalized vectors in ℓ_1 which converges pointwise to a normalized vector v, converges to v strongly.

LEMMA 5.3. *The series* $\sum_{x\in W} p_x x$ *and* $\sum_{x\in I} p_x x$ *are absolutely convergent.*

Proof. In fact, otherwise by (3.4), $\sum_{x\in W} p_x z^{**}(x)$ diverges to infinity. By (3.3), this divergence implies that $\sum_{x\in W} a_{x,i} z^{**}(x) \le C\|\sum_{x\in W} a_{x,i}x\| \to \infty$ as $i\to\infty$, contradicting the boundedness of $\{\sum_{x\in W} a_{x,i}x\}_{i=1}^{\infty}$. ■

LEMMA 5.4. *The series* $\sum_{y\in P} s_y y$ *is absolutely convergent, and thus so is* $\sum_{y\in P}(s_y - \sum_{\{x\in W:\,v(x)\succ v(y)\}} p_x)y$.

Proof. Assume the (first) absolute convergence does not hold. Since the vectors y are nonnegative and $s_y \ge 0$, by (3.4) the sum $\sum_y s_y z^{**}(y)$ diverges to ∞. On the other hand, for each $y\in P$ and sufficiently large $i = i(y)$ we have

$$z^{**}\Big(\sum_{\{x\in W:\,v(x)\succ v(y)\}} a_{x,i}x\Big) \ge \frac{1}{2}\, s_y z^{**}(y).$$

Since the sets $\{x\in W : v(x)\succ v(y)\}$ with different y are disjoint, by (3.3), we conclude that the sums $\sum_{x\in W} a_{x,i}x$ cannot be uniformly bounded.

The second statement is an immediate consequence of the first statement and the inequality $s_y \ge \sum_{\{x\in W: v(x)\succ v(y)\}} p_x$. This inequality is a consequence of Fatou's lemma for sequences:

$$\sum_{\{x\in W:\,v(x)\succ v(y)\}} p_x = \sum_{\{x\in W:\,v(x)\succ v(y)\}} \lim_{i\to\infty} a_{x,i} \le \lim_{i\to\infty} \sum_{\{x\in W:\,v(x)\succ v(y)\}} a_{x,i} = s_y. \ ■$$

LEMMA 5.5. *The sequence of vectors*

$$\sum_{x\in W} a_{x,i}x = \sum_{x\in I} a_{x,i}x + \sum_{y\in P}\sum_{\{x\in W:\,v(x)\succ v(y)\}} a_{x,i}x, \qquad i\in\mathbb{N}, \tag{5.7}$$

converges to the vector

$$\sum_{x\in I} p_x x + \sum_{y\in P}\Big(\sum_{\{x\in W:\,v(x)\succ v(y)\}} p_x x + \Big(s_y - \sum_{\{x\in W:\,v(x)\succ v(y)\}} p_x\Big)y\Big) \tag{5.8}$$

in the weak topology.*

Proof. We know that the vectors (5.7) are uniformly bounded. Hence it is enough to prove they converge to the vector (5.8) componentwise.

Let us consider their mth components. Assume that m is in the image of F_α and that the path to m from the initial vertex of the component containing m is $n_1, \dots, n_k = m$. There are two slightly different cases: $k = 1$ and $k \geq 2$. We consider the case $k \geq 2$; for $k = 1$ just replace n_{k-1} by n_1 in the formulas below and add the corresponding term for $x \in I$.

For $k \geq 2$ the mth component of the vector (5.8) is

$$p_z n_{k-1} + \sum_{\substack{y \in P \\ m \in \operatorname{supp} y}} s_y n_{k-1},$$

where in the first term, z is such that $m = v(z)$. On the other hand, the mth component of the vector (5.7) is

$$a_{z,i} n_{k-1} + \sum_{\substack{y \in P \\ m \in \operatorname{supp} y}} \sum_{\{x \in W : v(x) \succ v(y)\}} a_{x,i} n_{k-1},$$

where in the first term, z is such that $m = v(z)$.

We have

$$\sum_{\substack{y \in P \\ m \in \operatorname{supp} y}} \sum_{\{x \in W : v(x) \succ v(y)\}} a_{x,i} n_{k-1} \to \sum_{\substack{y \in P \\ m \in \operatorname{supp} y}} s_y n_{k-1}$$

by Lemma 5.2. ∎

To complete the proof of item (B), it suffices to show that the vector (5.8) is contained in the right-hand side of (5.5). To achieve this, we do the following:

(1) Recall that (see Lemma 5.1)

$$\sum_{x \in I} p_x + \sum_{y \in P} \sum_{\{x \in W : v(x) \succ v(y)\}} p_x + \sum_{y \in P} \Big(s_y - \sum_{\{x \in W : v(x) \succ v(y)\}} p_x \Big) = 1.$$

(2) Deduce from the previous item and Lemmas 5.3 and 5.4 that the vector in (5.8) is an infinite convergent convex combination of $x \in W$ and $y \in P$.

(3) Observe that $y \in P$ can appear in this combination with nonzero coefficient only if $\{x \in W : v(x) \succ v(y)\}$ is an infinite set. In fact, if this set is finite, then

$$\begin{aligned} &\sum_{\{x \in W : v(x) \succ v(y)\}} p_x \\ &\quad = \sum_{\{x \in W : v(x) \succ v(y)\}} \lim_{i \to \infty} a_{x,i} = \lim_{i \to \infty} \sum_{\{x \in W : v(x) \succ v(y)\}} a_{x,i} = s_y, \end{aligned}$$

and so the coefficient of y is 0.

(4) Prove that the infinite cardinality of $\{x \in W : v(x) \succ v(y)\}$ implies $y \in \bigcup_{0\le\gamma\le\beta+1} X^\gamma$. In fact, induction on ordinal η, $0 \le \eta \le \beta$, implies that finiteness of all $\{x \in X^\gamma : v(x) \succ v(y)\}$, $0 \le \gamma \le \eta$, implies

$$\{x \in X^\eta : v(x) \succ v(y)\} = \bigcup_{0\le\gamma\le\eta} \{x \in X^\gamma : v(x) \succ v(y)\}.$$

This leads to a contradiction for $\eta = \beta$. Hence, there is $\gamma \le \beta$ such that $\{x \in X^\gamma : v(x) \succ v(y)\}$ is infinite. So all these vertices are removed in the derived tree $F^{\gamma+1}$ and $y \in X^{\gamma+1}$ by definition.

5.4. Proof of item (C). Let $\beta < \alpha$; by Corollary 4.2, there is $y \in X^{\beta+1} \setminus \bigcup_{0\le\gamma\le\beta} X^\gamma$ such that the support of no vector in $\bigcup_{0\le\gamma\le\beta} X^\gamma$ is in $\operatorname{supp} y$. Let $m \in \mathbb{N}$ be the largest element of $\operatorname{supp} y$.

To complete the proof of (C) it suffices to show $y \notin \overline{\operatorname{conv}(\bigcup_{0\le\gamma\le\beta} X^\gamma)}$.

Combining the choice of y with the definition of X^γ, we find that the set $\bigcup_{0\le\gamma\le\beta} X^\gamma$ contains vectors of two types:

(1) Extensions of y, that is, vectors coinciding with y on its support, but also having at least one more positive coordinate. According to the definitions in Section 4, the coordinate has to be $\ge m$. We denote the set of all such vectors in $\bigcup_{0\le\gamma\le\beta} X^\gamma$ by E.
(2) Vectors whose mth coordinate is 0 and some coordinates not in $\operatorname{supp} y$ are positive. We denote the set of all such vectors in $\bigcup_{0\le\gamma\le\beta} X^\gamma$ by R.

Clearly,

$$\overline{\operatorname{conv}\Big(\bigcup_{0\le\gamma\le\beta} X^\gamma\Big)} = \overline{\operatorname{conv}(E \cup R)}.$$

So we need to find a continuous linear functional on $\mathcal{Z}^*$ which separates y from $\operatorname{conv}(E \cup R)$. In this connection, we consider the following two continuous linear functionals on $\mathcal{Z}^*$.

The first is the sum of all coordinates of a vector with respect to the basis $\{z_i^*\}_{i=1}^\infty$, except the coordinates which are nonzero for y. This functional is continuous because it is a linear combination of z^{**} (introduced in Section 3) and finitely many elements of $\{z_i\}_{i=1}^\infty$ considered as elements of $\mathcal{Z}^{**}$. We denote this functional by $\tilde z$. By definition, $y \in \ker(\tilde z)$.

The second functional is z_m (that is, the mth coordinate functional).

We claim that $z_m - \tilde z$ separates y from $\operatorname{conv}(E \cup R)$. To see this observe that $(z_m - \tilde z)(y) = a > 0$, where a is the value of the mth coordinate of y.

On the other hand, $(z_m - \tilde z)|_R \le 0$ because z_m is zero on R and $\tilde z$ is nonnegative for all vectors in R.

Also $(z_m - \tilde z)|_E \le 0$ because for each vector in the extension, further coordinates cannot be smaller than the previous ones.

5.5. Proof of item (D). To prove this statement, we repeat the argument used to prove (B) and observe that we get into the closure of the same set because, in this case, P is a subset of W. This is so for the following reason. By Lemma 2.1, derived trees of F_α stabilize at F_α^α, which is a set of isolated vertices. In terms of vectors of the set $X = X_\alpha$, this means that shortenings X^β of X stabilize when $\beta = \alpha$, and the resulting set X^α consists of vectors with 1-element support. Thus all vectors which can play the role of $y \in P$ are in $\bigcup_{0\le\gamma\le\alpha} X^\gamma$, which is the set W for the case under consideration, so in this case $P \subset W$.

The argument used in the proof of item (B) shows that $\overline{\mathrm{conv}(\bigcup_{0\le\gamma\le\alpha} X^\gamma)}$ is weak* sequentially closed. By the Krein–Šmulian theorem [10, p. 429], this completes the proof.

5.6. Proof of item (E). Let $\alpha = \kappa + 1$. By item (A), we have

$$A^{(\kappa)} \supset \mathrm{conv}\Big(\bigcup_{1\le\gamma\le\kappa} X^\gamma\Big).$$

Since the weak* derived set (of any set A) contains the strong closure of the set, we get

$$A^{(\kappa+1)} \supset \overline{\mathrm{conv}\Big(\bigcup_{1\le\gamma\le\kappa} X^\gamma\Big)}. \tag{5.9}$$

Let $k \in \mathbb{N}$ correspond to the initial vertex of the tree $F_{\kappa+1}$ according to the injective map constructed at the beginning of Section 4. Combining Lemma 2.1(2), the definition of $X^{\kappa+1}$, and the definition (4.1), we see that $X^{\kappa+1} = \{kz_k^*\}$. Let $r = kz_k^*$. Item (A) implies $r \in A^{(\kappa+1)}$. It remains to prove

$$A^{(\kappa+1)} \supset \overline{\mathrm{conv}\Big(\Big(\bigcup_{1\le\gamma\le\kappa} X^\gamma\Big) \cup \{r\}\Big)}.$$

Consider a strongly convergent sequence

$$\Big\{\sum_{x\in W} a_{x,i}x + a_{r,i}r\Big\}_{i=1}^\infty,$$

where $W = \bigcup_{1\le\gamma\le\kappa} X^\gamma$, $a_{x,i} \ge 0$, $a_{r,i} \ge 0$, and $\sum_{x\in W} a_{x,i} + a_{r,i} = 1$. Since $0 \le a_{r,i} \le 1$, we may assume that the sequence $\{a_{r,i}r\}_{i=1}^\infty$ is convergent. Since $A^{(\kappa+1)}$ is convex, by (5.9), the conclusion follows.

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