Universality and non-embeddability into Banach spaces of subspaces of the real line with the Gromov-Hausdorff distance

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Abstract

The paper aims to prove two universality results which can be used to simplify some of the available proofs of non-embeddability results for the Gromov-Hausdorff metrics.

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1 Introduction

The notion of the Hausdorff distance between two subsets of a metric space goes back to [7]. In [5], Gromov modified this notion and in this way created a tool for comparing two metric spaces not being subspaces of the same metric space. Gromov considered this modification utterly natural and, as such, continued to call it the Hausdorff distance. It should be pointed out that similar modifications were introduced by, for example, Edwards [3], Kadets [10] and, possibly, many others. Meanwhile, it was Gromov [5] who developed a way of thinking based on this generalization and demonstrated its first impressive applications. For this reason, it became customary to call this distance the *Gromov-Hausdorff* distance. We follow this tradition.

To present the results of this work, let us remind the necessary definitions. The *Hausdorff distance* $d_H(X,Y)$ between two subsets X and Y of a metric spaces Z is defined as the infimum of all $\varepsilon > 0$ such that Y is in the ε-neighborhood of X and X is in the ε-neighborhood of Y. Notice that $d_H(X,Y)$ can be infinite. The *Gromov-Hausdorff* distance between metric spaces X and Y is defined as $\inf_Z d_H(X,Y)$, where the infimum is taken over all metric spaces Z, such that $Z = X \cup Y$ as a set, and the metric on Z agrees with the metric of X on X and the metric of Y on Y.

For a general introduction to the Gromov-Hausdorff distance we refer to [1]. It is worth mentioning that nowadays the arguments in the geometric group theory, based on the Gromov-Hausdorff distance, can be expressed in terms of asymptotic cones, see [6, Section 2]. The Gromov-Hausdorff distance is also used in Functional Analysis, see [11]. In addition, the Gromov-Hausdorff distance has gained a popularity in applied areas, such as *shape recognition* and *cluster analysis*. See [2, 14, 15] and references therein.

Recently, the investigation of the bilipschitz and coarse embeddability for some of the Gromov-Hausdorff metrics - in the sense of the Metric Embeddings theory, see [16] - into Banach spaces with some special properties has been initiated in a number of researches. See [17], [18] and references therein. The present article fits this area.

Our aim is to prove two universality results that allow to simplify significantly proofs of some available results in this direction. For example, applying Theorem 1 one can derive a simpler proof of Zava's result given in [18, Theorem 4.2].

2 Statement of results

Consider finite subsets of $\mathbb{R}$ as metric spaces with the induced metrics. Denote by $\mathbf{Fin}(\mathbb{R})$ the set of all such finite metric spaces. We endow this set of metric spaces with the Gromov-Hausdorff distance. Our first universality result is:

Theorem 1. *Each finite metric space can be isometrically embedded into the metric space* $(\mathbf{Fin}(\mathbb{R}), d_{GH})$.

In order to proceed, let us recall some definitions.

Definition 1. A metric space (A,d) is *bilipschitz embeddable* into a Banach space X with distortion $C \geq 1$ is there is a map $F : A \to X$ such that $\forall u, v \in A \quad d(u,v) \leq \|F(u) - F(v)\| \leq Cd(u,v)$.

A metric space (A,d) is *coarsely embeddable* into X with (non-negative, indefinitely increasing at ∞ on $[0,\infty)$) *control functions* ρ_1, ρ_2 if there is a map $F : A \to X$ such that $\forall u, v \in A \quad \rho_1(d(u,v)) \leq \|F(u) - F(v)\| \leq \rho_2(d(u,v))$.

According to these definitions, the next immediate consequence of Theorem 1 can be stated as follows:

Corollary 1. *If there exists a sequence of finite metric spaces which do not admit embeddings into some Banach space X with uniformly bounded distortions/with the same control functions for the whole sequence, then* $(\mathbf{Fin}(\mathbb{R}), d_{GH})$ *does not admit a bilipschitz embedding into X/does not admit a coarse embedding into X.*

The goal of [18, Theorem 4.2] is to prove a version of Corollary 1. The argument in [18] is based on a different notion of universality. The proof is substantially more complicated than that of Theorem 1 and allows to prove a version of Corollary 1 for coarse embeddings only. As in [18], we combine a universality result with a non-embeddability one for finite metric spaces. Zava [18] applied the result of Lafforgue [12] stating that there exist a sequence of finite metric spaces which do not admit coarse embeddings into a uniformly convex Banach space with the same control functions.

Note that there are numerous other results to which Corollary 1 is applicable. Some of them can be found in [12], [13], and the monograph [16].

A metric space (A, d) is called *uniformly discrete*, if there exists a real number $\delta > 0$ such that $d(u, v) \geq \delta$ for any two elements $u, v \in A$. Denote by $\mathbf{UD}(\mathbb{R})$ the set of all uniformly discrete subsets of the real line $\mathbb{R}$, considered as metric spaces with metrics induced from $\mathbb{R}$. We endow this set of metric spaces with the Gromov-Hausdorff distance.

Our second universality result is the following.

Theorem 2. *Each metric space of finite diameter having a countable dense subset admits an isometric embedding into* $(\mathbf{UD}(\mathbb{R}), d_{GH})$ *.*

Note that every compact metric space satisfies the condition of Theorem 2.

Remark 1. *The statements of our universality results sound similar to the statements of the results in [8] and [9]. Nevertheless, our and their results are quite different. In distinction to the results of Theorems 1 and 2, the target space in the mentioned papers is the space of isometry classes of all compact metric spaces with the Gromov-Hausdorff distance.*

3 Proofs of the results

Proof of Theorem 1. Let (M, d) be a finite metric space whose elements are $\{x_1, \ldots, x_n\}$ and let D be the diameter of this set. We introduce a map $S : M \to (\mathbf{Fin}(\mathbb{R}), d_{GH})$ as described below:

The image $S(x_i) =: s_i$ is a subset of $\mathbb{R}$, whose $2n + 1$ elements - listed in the increasing order - are $\{m_k(x_i)\}_{k=0}^{2n}$. These elements are given by:

- $m_0(x_i) = -5D$ (for all elements of M)
- $m_1(x_i) = 3D + d(x_i, x_1)$
- $m_2(x_i) = 7D - d(x_i, x_1)$
- $m_3(x_i) = 9D + d(x_i, x_2)$
- $m_4(x_i) = 13D - d(x_i, x_2)$
- $\ldots$
- $m_{2n-1}(x_i) = 3(2n - 1)D + d(x_i, x_n)$
- $m_{2n}(x_i) = 3(2n)D + D - d(x_i, x_n)$

Note that each set s_i is of the following form:

- It starts with a point $-5D$
- The next point is in the interval $[3D, 4D]$
- The next point is in the interval $[6D, 7D]$
- $\ldots$

- The last point is in the interval $[3(2n)D, (3(2n)+1)D]$

Now, we need the following auxiliary results:

Lemma 1. *$d_{GH}(s_i, s_j) \leq d(x_i, x_j)$.*

Proof. To prove this inequality it suffices to find the Hausdorff distance between s_i and s_j as subsets of $\mathbb{R}$. The triangle inequality implies $d_H(s_i, s_j) \leq \max_k |d(x_i, x_k) - d(x_j, x_k)| = d(x_i, x_j)$ and the statement follows. □

The next observation is going to come in handy.

Observation 1. *If $I_i : s_i \to \mathcal{U}$ and $I_j : s_j \to \mathcal{U}$ are two isometric embeddings into a metric space $(\mathcal{U}, d_{\mathcal{U}})$, and the Hausdorff distance d_H between the images satisfies $d_H < D$, then $d_{\mathcal{U}}(I_i(m_k(x_i)), I_j(m_k(x_j))) \leq d_H$ for every $k = 0, \ldots, 2n$.*

Proof. Assume the contrary, and let k_0 be the smallest k for which

$$d_{\mathcal{U}}(I_i(m_k(x_i)), I_j(m_k(x_j))) > d_H.$$

First, we consider the case $k_0 = 0$. In this case, there exists $t > 0$ such that $d_{\mathcal{U}}(I_i(m_0(x_i)), I_j(m_t(x_j))) \leq d_H < D$. This implies that

$$2D - d_H \leq d_{\mathcal{U}}(I_i(m_0(x_i), I_j(m_{t\pm1}(x_j))) \leq 4D + d_H \tag{1}$$

for those of $\{t-1, t+1\}$ which are in the set $\{1, \ldots, 2n\}$.

On the other hand, since I_i is an isometry, for each $k \in \{1, \ldots, 2n\}$, the inequality below is true:

$$d_{\mathcal{U}}(I_i(m_0(x_i), I_i(m_k(x_i))) \geq 8D.$$

Thus, none of the elements $I_i(m_k(x_i)), k \in \{0, \ldots, 2n\}$, can be within distance $d_H \leq D$ from $I_j(m_{t\pm1}(x_j))$.

This contradiction with our assumption on the Gromov-Hausdorff distance yields:

$$d_{\mathcal{U}}(I_i(m_0(x_i)), I_j(m_0(x_j))) \leq d_H.$$

Since I_i and I_j are isometries, we get

$$5 + 3kD \leq d_{\mathcal{U}}(I_i(m_0(x_i)), I_i(m_k(x_i))) \leq 5 + (3k+1)D,$$

$$5 + 3kD \leq d_{\mathcal{U}}(I_j(m_0(x_j)), I_j(m_k(x_j))) \leq 5 + (3k+1)D,$$

for every $k \in \{1, 2, \ldots, 2n\}$.

Combining these inequalities with the fact that $d_H < D$ and the triangle inequality, one concludes that $I_i(m_k(x_i))$ is the only element of the sequence $\{I_i(m_k(x_i))\}_{k=0}^{2n+1}$ which can be within distance $d_H < D$ from $I_j(m_k(x_j))$. More details for this argument can be found in the last part of the proof of Observation 2. There, the statement is established for infinite sequences, but the finite version of that proof is immediate.

The observation is proved. □

Lemma 2. *$d_{GH}(s_i, s_j) \geq d(x_i, x_j)$.*

Proof. Observation 1 implies that if the Hausdorff distance d_H between the images of some isometric embeddings $I_i; s_i \to \mathcal{U}$ and $I_j : s_j \to \mathcal{U}$ satisfies $d_H < d(x_i, x_j)$, then

$$d_{\mathcal{U}}(I_i(m_k(x_i)), I_j(m_k(x_j))) < d(x_i, x_j) \quad \text{for all } k. \tag{2}$$

To derive a contradiction, we consider the following four points:
$m_{2i-1}(x_i) = 3(2i-1)D$,
$m_{2i}(x_i) = 3(2i)D + D$,
$m_{2i-1}(x_j) = 3(2i-1)D + d(x_j, x_i)$,
$m_{2i}(x_j) = 3(2i)D + D - d(x_j, x_i)$.
Since I_i and I_j are isometries, the equalities below hold:

$$d_{\mathcal{U}}(I_i(m_{2i-1}(x_i)), I_i(m_{2i}(x_i))) = 4D.$$

$$d_{\mathcal{U}}(I_j(m_{2i-1}(x_j)), I_j(m_{2i}(x_j))) = 4D - 2d(x_i, x_j).$$

This leads to a contradiction with (2) since

$$\begin{aligned} 4D &= d_{\mathcal{U}}(I_i(m_{2i-1}(x_i)), I_i(m_{2i}(x_i))) \\ &\leq d_{\mathcal{U}}(I_i(m_{2i-1}(x_i)), I_j(m_{2i-1}(x_j))) + d_{\mathcal{U}}(I_j(m_{2i-1}(x_j)), I_j(m_{2i}(x_j))) \\ &\qquad + d_{\mathcal{U}}(I_j(m_{2i}(x_j)), I_i(m_{2i}(x_i))) \\ &\overset{(2)}{<} d(x_i, x_j) + (4D - 2d(x_i, x_j)) + d(x_i, x_j) = 4D. \end{aligned}$$ □

□

Proof of Theorem 2. This proof goes along the same lines as our proof of Theorem 1. For the sake of convenience, we change some notation.

Let (M, d) be a metric space of finite diameter D, having a countable dense subset $\{x_i\}_{i=1}^{\infty}$.

We introduce an embedding of $S : M \to (\mathbf{UD}(\mathbb{R}), d_{GH})$ by $S(x) = \{m_i(x)\}_{i=0}^{\infty}$, where $\{m_i(x)\}_{i=0}^{\infty}$ is an increasing sequence in $\mathbb{R}$ given by

- $m_0(x) = -5D$ (for all x)
- $m_1(x) = 3D + d(x, x_1)$
- $m_2(x) = 7D - d(x, x_1)$
- $m_3(x) = 9D + d(x, x_2)$
- $m_4(x) = 13D - d(x, x_2)$
- ...
- $m_{2i-1}(x) = 3(2i-1)D + d(x, x_i)$
- $m_{2i}(x) = 3(2i)D + D - d(x, x_i)$
- ...

The sequence $\{m_i(x)\}_{i=0}^{\infty}$ is uniformly discrete for every x because it satisfies the conditions:

- $m_0(x) = -5D$

- The point $m_1(x)$ is in the interval $[3D, 4D]$
- The point $m_2(x)$ is in the interval $[6D, 7D]$
- ...
- The point $m_i(x)$ is in the interval $[3iD, (3i+1)D]$
- ...

At this point, we need the next

Lemma 3. *For any $x, y \in M$ $d_{GH}(S(x), S(y)) \leq d(x, y)$.*

Proof. To get an upper estimate, it suffices to evaluate the Hausdorff distance between $S(x)$ and $S(y)$ as subsets in $\mathbb{R}$. The triangle inequality implies

$$d_H(S(x), S(y)) \leq \max_i |d(x, x_i) - d(y, x_i)| \leq d(x, y)$$

and the lemma follows. □

Observation 2. *If $I_x : S(x) \to \mathcal{U}$ and $I_y : S(y) \to \mathcal{U}$ are two isometric embeddings into a metric space $(\mathcal{U}, d_{\mathcal{U}})$, and the Hausdorff distance d_H between the images satisfies $d_H < D$, then $d_{\mathcal{U}}(I_x(m_i(x)), I_y(m_i(y))) \leq d_H$ for every $i \in \{0\} \cup \mathbb{N}$.*

Proof. Assume the contrary, and let i_0 be the smallest i for which

$$d_{\mathcal{U}}(I_x(m_i(x)), I_y(m_i(y))) > d_H.$$

First, consider the case $i_0 = 0$. In this case, for some $t > 0$, there holds:

$$d_{\mathcal{U}}(I_x(m_0(x)), I_y(m_t(y))) \leq d_H < D.$$

If $t > 1$, this implies that $2D - d_H \leq d_{\mathcal{U}}(I_x(m_0(x)), I_y(m_{t\pm1}(y))) \leq 4D + d_H$ while if $t = 1$, the inequality

$$2D - d_H \leq d_{\mathcal{U}}(I_x(m_0(x)), I_y(m_{t\pm1}(y))) \leq 4D + d_H \tag{3}$$

is guaranteed only for $t + 1$.

On the other hand, since I_x is an isometry, we have

$$d_{\mathcal{U}}(I_x(m_0(x)), I_x(m_k(x))) \geq 8D \quad \text{for each } k \in \mathbb{N}.$$

Consequently, none of the elements $I_x(m_k(x)), k \in \{0\} \cup \mathbb{N}$, can be within distance $d_H \leq D$ from $I_y(m_{t+1}(y))$.

This contradiction implies that $d_{\mathcal{U}}(I_x(m_0(x)), I_y(m_0(y))) \leq d_H$. Since I_x and I_y are isometries, one gets:

$$5 + 3iD \leq d_{\mathcal{U}}(I_x(m_0(x)), I_y(m_i(x))) \leq 5 + (3i+1)D,$$

$$5 + 3iD \leq d_{\mathcal{U}}(I_y(m_0(y)), I_y(m_i(y))) \leq 5 + (3i+1)D.$$

At this stage, our goal is to show that these inequalities imply that we have

$$d_{\mathcal{U}}(I_x(m_i(x)), I_y(m_j(y))) > D \quad \text{for each } i, j \in \mathbb{N} \quad \text{with } i \neq j$$

and, therefore, $I_y(m_i(y))$ is the only element of the sequence $\{I_y(m_k(y))\}_{k=0}^{\infty}$ which can be within distance $d_H < D$ from $I_x(m_i(x))$.

To reach this goal, assume without loss of generality that $i < j$. Then,

$$\begin{aligned}
&d_{\mathcal{U}}(I_x(m_i(x)), I_y(m_j(y))) \\
&\quad \geq d_{\mathcal{U}}(I_y(m_j(y)), I_y(m_0(y))) - d_{\mathcal{U}}(I_y(m_0(y)), I_x(m_0(x))) - d_{\mathcal{U}}(I_x(m_0(x)), I_x(m_i(x))) \\
&\quad \geq 5 + 3jD - d_H - (5 + (3i+1)D) = (3j - 3i - 1)D - d_H \geq 2D - d_H > D.
\end{aligned}$$

This contradiction proves the observation. □

To finalize the proof of the theorem, one more lemma is needed.

Lemma 4. *For any $x, y \in X$, the inequality $d_{GH}(S(x), S(y)) \geq d(x, y)$ holds.*

Proof. Assume the contrary, that is, suppose that there exist $x, y \in X$, $\varepsilon > 0$, and embeddings $I_x : S(x) \to \mathcal{U}$ and $I_y : S(y) \to \mathcal{U}$, such that $d_H(S(x), S(y)) < d(x, y) - \varepsilon$. By Observation 2, this implies that $d_{\mathcal{U}}(I_x(m_i(x)), I_y(m_i(y))) < d(x, y) - \varepsilon$ for every $i \in \{0\} \cup \mathbb{N}$. Since $\{x_k\}_{k=1}^{\infty}$ is dense in M, one can pick $i \in \mathbb{N}$ so that $d(x, x_i) < \varepsilon/2$.

To derive a contradiction, consider the following four points on the real line:
$m_{2i-1}(x) = 3(2i-1)D + d(x, x_i) < 3(2i-1)D + \frac{\varepsilon}{2}$,
$m_{2i}(x) = 3(2i)D + D - d(x, x_i) > 3(2i)D + D - \frac{\varepsilon}{2}$,
$m_{2i-1}(y) = 3(2i-1)D + d(y, x_i) > 3(2i-1)D + d(x, y) - \frac{\varepsilon}{2}$,
$m_{2i}(y) = 3(2i)D + D - d(y, x_i) < 3(2i)D + D - d(x, y) + \frac{\varepsilon}{2}$.
Since I_x and I_y are isometries, the inequalities above amount to

$$d_{\mathcal{U}}(I_x(m_{2i}(x)), I_x(m_{2i-1}(x))) > 4D - \varepsilon.$$

$$d_{\mathcal{U}}(I_y(m_{2i}(y)), I_y(m_{2i-1}(y))) < 4D - 2d(x, y) + \varepsilon.$$

This leads to a contradiction because we have:

$$\begin{aligned}
4D &< d_{\mathcal{U}}(I_x(m_{2i}(x)), I_x(m_{2i-1}(x))) + \varepsilon \\
&< d_{\mathcal{U}}(I_x(m_{2i}(x)), I_y(m_{2i}(y))) + d_{\mathcal{U}}(I_y(m_{2i}(y)), I_y(m_{2i-1}(y))) \\
&\quad + d_{\mathcal{U}}(I_y(m_{2i-1}(y)), I_x(m_{2i-1}(x))) + \varepsilon \\
&< d(x, y) - \varepsilon + 4D - 2d(x, y) + \varepsilon + d(x, y) - \varepsilon + \varepsilon = 4D.
\end{aligned}$$ □

Theorem 2 now follows from Lemmas 3 and 4. □

Remark 2. *Our embeddings in Theorems 1 and 2 can be regarded as pointed two-sided Fréchet embeddings, see [4] and [16, Proposition 1.17].*

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