ANALYSIS ON LAAKSO GRAPHS WITH APPLICATION TO THE STRUCTURE OF TRANSPORTATION COST SPACES

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Abstract. This article is a continuation of our article in [Canad. J. Math. Vol. 72 (3), (2020), pp. 774–804]. We construct orthogonal bases of the cycle and cut spaces of the Laakso graph $\mathcal{L}_n$. They are used to analyze projections from the edge space onto the cycle space and to obtain reasonably sharp estimates of the projection constant of $\mathrm{Lip}_0(\mathcal{L}_n)$, the space of Lipschitz functions on $\mathcal{L}_n$. We deduce that the Banach-Mazur distance from $\mathrm{TC}(\mathcal{L}_n)$, the transportation cost space of $\mathcal{L}_n$, to ℓ_1^N of the same dimension is at least $(3n-5)/8$, which is the analogue of a result from [op. cit.] for the diamond graph D_n. We calculate the exact projection constants of $\mathrm{Lip}_0(D_{n,k})$, where $D_{n,k}$ is the diamond graph of branching k. We also provide simple examples of finite metric spaces, transportation cost spaces on which contain ℓ_∞^3 and ℓ_∞^4 isometrically.

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1. Introduction

1.1. **Definitions and background.** Let (X,d) be a metric space. Consider a real-valued finitely supported function f on X with a zero sum, that is, $\sum_{v\in \operatorname{supp} f} f(v)=0$. A natural and important interpretation of such a function, is considering it as a *transportation problem*: one needs to transport certain product from locations where $f(v)>0$ to locations where $f(v)<0$.

One can easily see that f can be represented as

$$f = a_1(\mathbf{1}_{x_1}-\mathbf{1}_{y_1}) + a_2(\mathbf{1}_{x_2}-\mathbf{1}_{y_2}) + \cdots + a_n(\mathbf{1}_{x_n}-\mathbf{1}_{y_n}), \qquad (1)$$

where $a_i\ge 0$, $x_i,y_i\in X$, and $\mathbf{1}_u(x)$ for $u\in X$ is the *indicator function* of u, defined by

$$\mathbf{1}_u(x)=\begin{cases}1 & \text{if } x=u,\\ 0 & \text{if } x\ne u.\end{cases}$$

We call each such representation a *transportation plan* for f, and it can be interpreted as a plan of moving a_i units of the product from x_i to y_i. The *cost* of the transportation plan (1) is defined as $\sum_{i=1}^n a_i d(x_i,y_i)$.

Remark 1. It is worth mentioning that in our discussion transportation plans are allowed to be *fake plans*, in the sense that it can happen that there is no product in x_i in order to make the delivery to y_i. To see what we mean consider a metric space containing three distinct points x, y, z. Then $(\mathbf{1}_x - \mathbf{1}_y) + (\mathbf{1}_y - \mathbf{1}_z) + (\mathbf{1}_z - \mathbf{1}_x)$ is a transportation plan for function 0 (null transportation problem, nothing is needed or available), although there is no product in x to be delivered to y. However, it is easy to show that the defined below *optimal transportation plans* can be implemented.

We denote the real vector space of all transportation problems by $\mathrm{TP}(X)$. We introduce the *transportation cost norm* (or just *transportation cost*) $\|f\|_{\mathrm{TC}}$ of a transportation problem f as the infimum of costs of transportation plans satisfying (1). Using the triangle inequality and compactness it is easy to show that the infimum of costs of transportation plans for f is attained. A transportation plan for f whose cost is equal to $\|f\|_{\mathrm{TC}}$ is called an *optimal transportation plan.* The completion of the normed space $(\mathrm{TP}(X), \|\cdot\|_{\mathrm{TC}})$ is called a *transportation cost space* and is denoted by $\mathrm{TC}(X)$.

We use the standard terminology of Banach space theory [4], graph theory [7], and the theory of metric embeddings [25].

Transportation cost spaces are of interest in many areas and are studied under many different names (we list some of them in the alphabetical order: Arens-Eells space, earth mover distance, Kantorovich-Rubinstein distance, Lipschitz-free space, Wasserstein distance). We prefer to use the term *transportation cost space* since it makes the subject of this work instantly clear to a wide circle of readers and it also reflects the historical approach leading to these notions (see [15, 16]). Interested readers can find a review of the main definitions, notions, facts, terminology and historical notes pertinent to the subject in [22, Section 1.6].

By a *pointed metric space* we mean a metric space (X, d_X) with a *base point*, denoted by O. For a pointed metric space X with a base point at O by $\mathrm{Lip}_0(X)$ we denote the space of all Lipschitz functions $f : X \to \mathbb{R}$ satisfying $f(O) = 0$. It is not difficult to check that $\mathrm{Lip}_0(X)$ is a Banach space with respect to the norm $\|f\| = \mathrm{Lip}(f)$ ($\mathrm{Lip}(f)$ is the Lipschitz constant of f). As is well known $\mathrm{TC}(X)^* = \mathrm{Lip}_0(X)$ (see e.g. [25, Section 10.2]).

One of the main goals of this paper is to study the geometry of the spaces $\mathrm{TC}(X)$. We are interested mostly in the case where X is finite. We would like to mention that for finite X, the space $\mathrm{TC}(X)$ is an ℓ_1-like space in the sense that is has three qualities which make it close to $\ell_1^{|X|-1}$.

(1) It has a 1-complemented subspace isometric to $\ell_1^{\lceil |X|/2 \rceil}$, see [17] (a weaker version was proved earlier in [8]).

(2) It admits a linear embedding into $L_1[0, 1]$ with distortion $\le C \ln |X|$, see [5, 9, 13]. Although this result is known since 2003, it seems that the only source where one can find its published proof is [3, Theorem 15].

(3) It is a quotient of ℓ_1^d with $d \le |X|^2$, see [23]. Another proof and a more precise statement can be found in Section 7.

However, $\mathrm{TC}(X)$ is isometric to $\ell_1^{|X|-1}$ if and only if X is a weighted tree. This result can be derived from the general result of [6]. Apparently the finite case of this result can be considered as folklore, for convenience of the readers we give a direct proof of the "only if" part (for finite case) in Section 7, the "if" part can be found in [8, Proposition 2.1].

One of the important problems about transportation cost spaces is the following [8, Problem 2.6]:

Problem 2. *It would be very interesting to find a condition on a finite metric space M which is equivalent to the condition that the space $\mathrm{TC}(M)$ is Banach-Mazur close to ℓ_1^n of the corresponding dimension. It is not clear whether it is feasible to find such a condition.*

In [8] we investigated this problem for large recursive families of graphs which include well-known families of diamond and Laakso graphs.

The main goal of this paper is further development of analysis in the space of functions on diamond and Laakso graphs in order to sharpen results of [8]. Let us remind the definitions of these families of graphs.

Definition 3 (Diamond graphs)**.** Diamond graphs $\{D_n\}_{n=0}^\infty$ are defined recursively: The *diamond graph* of level 0 has two vertices joined by an edge of length 1 and is denoted by D_0. The *diamond graph* D_n is obtained from D_{n-1} in the following way. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. (See Figure 1.)

Apparently Definition 3 was first introduced in [12].

Let us count some parameters associated with the graphs D_n. Denote by $V(D_n)$ and $E(D_n)$ the vertex set and edge set of D_n, respectively. Note that:

(a) $|E(D_n)| = 4^n$.
(b) $|V(D_{n+1})| = |V(D_n)| + 2|E(D_n)|$.

Hence $|V(D_n)| = 2(1 + \sum_{i=0}^{n-1} 4^i)$.

Definition 4 (Multibranching diamonds)**.** For any integer $k \ge 2$, we define $D_{0,k}$ to be the graph consisting of two vertices joined by one edge. For any $n \in \mathbb{N}$, if the graph $D_{n-1,k}$ is already defined, the graph $D_{n,k}$ is defined as the graph obtained from $D_{n-1,k}$ by replacing each edge uv in $D_{n-1,k}$ by a set of k independent paths of length 2 joining u and v. We endow $D_{n,k}$ with the shortest path distance. We call $\{D_{n,k}\}_{n=0}^\infty$ *diamond graphs of branching* k, or *diamonds of branching* k.

Definition 4 was introduced in [20]. Note that:

(a) $|E(D_{n,k})| = (2k)^n$.
(b) $|V(D_{n+1,k})| = |V(D_{n,k})| + k|E(D_{n,k})|$.

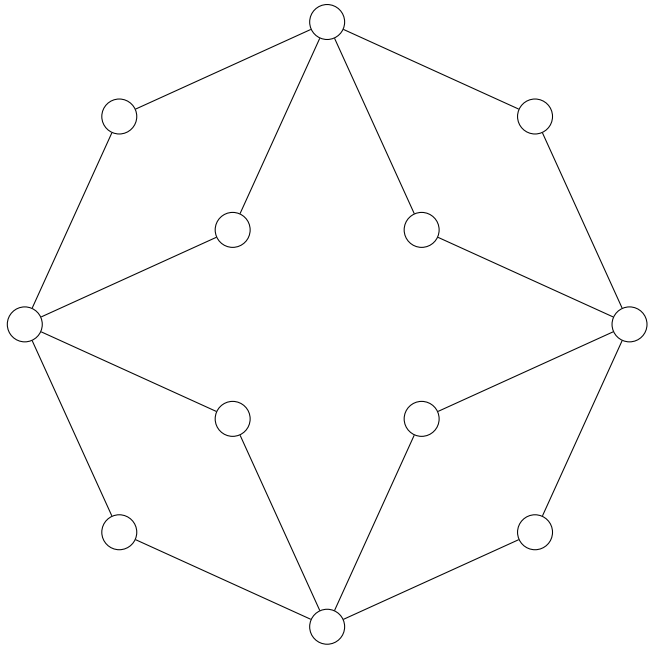

FIGURE 1. Diamond D_2.

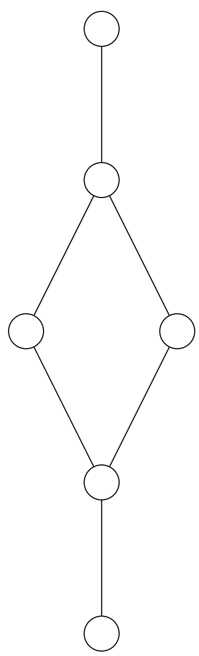

FIGURE 2. Laakso graph $\mathcal{L}_1$.

Hence $|V(D_{n,k})| = 2 + k\sum_{i=0}^{n-1}(2k)^i$.

Definition 5. Laakso graphs $\{\mathcal{L}_n\}_{n=0}^{\infty}$ are defined recursively: The *Laakso graph* of level 0 has two vertices joined by an edge of length 1 and is denoted $\mathcal{L}_0$. The *Laakso graph* $\mathcal{L}_n$ is obtained from $\mathcal{L}_{n-1}$ according to the following procedure. Each edge $uv \in E(\mathcal{L}_{n-1})$ is replaced by the graph $\mathcal{L}_1$ exhibited in Figure 2, the vertices u and v are identified with the vertices of degree 1 of $\mathcal{L}_1$.

Definition 5 was introduced in [19], where an idea of Laakso [18] was used. Note that:

(a) $|E(\mathcal{L}_n)| = 6^n$.
(b) $|V(\mathcal{L}_{n+1})| = |V(\mathcal{L}_n)| + 4|E(\mathcal{L}_n)|$.

Hence $|V(\mathcal{L}_n)| = 2 + 4\sum_{i=0}^{n-1} 6^i$.

Diamond and Laakso graphs play important roles in Metric Geometry as examples/counterexamples to many natural questions. One of the reasons for interest in the families of graphs introduced in Definitions 3-5 is that their bilipschitz embeddability characterizes non-superreflexive Banach spaces [14, 24, 26]. In [21] it was shown that Laakso graphs are incomparable with diamond graphs in the following sense: elements of none of these families admit bilipschitz embeddings into the other family with uniformly bounded distortions.

We need the following description of TC(X) in the case where X is a vertex set of an unweighted graph with its graph distance. Let $G = (V(G), E(G)) = (V, E)$ be a finite graph. Let $\ell_1(E)$, $\ell_2(E)$, and $\ell_\infty(E)$ be the spaces of real-valued functions on E with the norms $\|f\|_1 = \sum_{e\in E}|f(e)|$, $\|f\|_2 = \left(\sum_{e\in E}|f(e)|^2\right)^{\frac{1}{2}}$, and $\|f\|_\infty = \max_{e\in E}|f(e)|$, respectively. We also consider the inner product $\langle f, g\rangle$ associated with $\|f\|_2$.

We consider an arbitrary chosen orientation on E, so each edge of E is a directed edge. We denote by e^+ and e^- the *head* and *tail* of an oriented edge e, respectively. The choice of orientation affects some of the objects which we introduce, but does not affect the final results. Such orientation is usually called *reference orientation*.

For a directed cycle C in E (we mean that the cycle can be "walked around" following the direction, which is not related with the orientation of E) we introduce the *signed indicator function* of C by

(2)
$$\chi_C(e) = \begin{cases} 1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ are the same} \\ -1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ are different} \\ 0 & \text{if } e \notin C. \end{cases}$$

The *cycle space* $Z(G)$ of G is the subspace of $\ell_1(E)$ spanned by the signed indicator functions of all cycles in G. The orthogonal complement of $Z(G)$ in $\ell_2(E)$ is called the *cut space*.

We will use the fact ([25, Proposition 10.10]) that $\mathrm{TC}(G)$ for unweighted graphs G is isometrically isomorphic to the quotient of $\ell_1(E)$ over $Z(G)$:

$$\mathrm{TC}(G) = \ell_1(E)/Z(G) \tag{3}$$

The paper [23] contains a generalization of (3) for weighted graphs, and thus for arbitrary finite metric spaces.

For convenience of the readers we give a simple proof of (3).

Proof. Observe that if $G = (V, E)$ is endowed with a reference orientation, each function $f \in \ell_1(E)$ can be regarded as transportation plan given by

$$\sum_{e\in E} f(e)(\mathbf{1}_{e^-} - \mathbf{1}_{e^+}),$$

and the cost of this plan is $\|f\|_1$ (note that $f(e)$ can be negative, so this transportation plan is not necessarily in the form (1)).

In turn, each such transportation plan gives (after summation) the transportation problem which it solves. Thus (for any fixed reference orientation) there is a natural linear map $T : \ell_1(E) \to \mathrm{TP}(G) = \mathrm{TC}(G)$ (we consider finite graphs). The statement in the previous paragraph implies that $\|Tf\|_{\mathrm{TC}} \le \|f\|_1$.

It remains to show that for each transportation problem $x \in \mathrm{TC}(G)$ there is $f \in \ell_1(E)$, such that $Tf = x$ and $\|f\|_1 = \|x\|_{\mathrm{TC}}$.

Let $\sum_{i=1}^n a_i(\mathbf{1}_{x_i} - \mathbf{1}_{y_i})$ be an optimal transportation plan for x. Since pairs x_iy_i do not necessarily form edges, this optimal transportation plan does not immediately and naturally correspond to a vector in $\ell_1(E)$. Nevertheless, by the definition of a graph distance, for each such pair x_iy_i, we can find a shortest path $u_{0,i}, u_{1,i}, \dots, u_{m(i),i}$ in G with $u_{0,i} = x_i$, $u_{m(i),i} = y_i$, each pair $u_{j-1,i}u_{j,i}$ ($j = 1, \dots, m(i)$) being an edge in G, and $m(i) = d(x_i, y_i)$.

Then, as is easy to see,

$$\sum_{i=1}^{n}\sum_{j=1}^{m(i)} a_i(\mathbf{1}_{u_{j-1,i}} - \mathbf{1}_{u_{j,i}}),$$

is also an optimal transportation plan for x and this plan corresponds to a vector f in $\ell_1(E)$ with $\|f\|_1 = \|x\|_{\mathrm{TC}}$.

The correspondence is the following: $f(e) = 0$ is e is not of the form $u_{j-1,i}u_{j,i}$ for some i and j, and $f(e) = \theta(e, i, j)a_i$, if e is of the form $u_{j-1,i}u_{j,i}$, where $\theta(e, i, j) = 1$ if $u_{j-1,i}$ is the tail of e and $\theta(e, i, j) = -1$ if $u_{j-1,i}$ is the head of e. □

1.2. **Results from** [8] **on iteratively defined graphs.** Let us recall two results from [8] which are relevant to the present work.

A directed graph B having two distinguished vertices which we call *top* and *bottom*, generates a recursive family $\{B_n\}_{n=0}^{\infty}$ as follows:

- The graph B_0 consists of one directed edge.

- For $n \geq 1$, B_n is obtained from B_{n-1} by replacing each edge by a copy of B, identifying bottom of B with the tail of the edge and top of B with the head of the edge. Edges of B_n inherit their directions from the corresponding copies of B.

In [8] we considered the recursive families corresponding to directed graphs B satisfying certain natural conditions listed in [8, Section 4.1]), which include the multibranching diamond and Laakso graphs defined above.

Theorem A. [8, Theorem 4.2] *If the directed graph B satisfies the conditions of* [8, Section 4.1] *and $\{B_n\}_{n=0}^{\infty}$ is the corresponding recursively defined family then the Banach-Mazur distance to $\ell_1^{d(n)}$ satisfies*

$$d_{BM}(\mathrm{TC}(B_n), \ell_1^{d(n)}) \geq \frac{cn}{\ln n}$$

for $n \geq 2$ and some absolute constant $c > 0$, where $d(n)$ is the dimension of $\mathrm{TC}(B_n)$.

The $\ln n$ factor in Theorem A was removed for the case of multibranching diamond graphs and an upper bound was also proved.

Theorem B. [8, Theorem 6.10] *The Banach-Mazur distance $d_{n,k}$ from the transportation cost space* $\mathrm{TC}(D_{n,k})$ *to the ℓ_1^N space of the same dimension satisfies*

$$4n + 4 \geq d_{n,k} \geq \frac{k-1}{2k} n.$$

1.3. **Statement of results.** Our main goal is to investigate the analogue of Theorem B for the Laakso graph $\mathcal{L}_n$. In Section 5 we prove the lower bound of $(3n-5)/8$ for the Banach-Mazur distance from $\mathrm{TC}(\mathcal{L}_n)$ to ℓ_1^N (Corollary 17). This removes the $\ln n$ factor of Theorem A and is the analogue of the lower bound in Theorem B. However, we have not succeeded in proving a comparable (e.g. $O(n^a)$) upper bound. The obstacle to proving an analogue of the upper bound in Theorem B is explained in Section 7.

Our analysis of $\mathrm{TC}(\mathcal{L}_n)$ is based on the fact (see (3)) that $\mathrm{TC}(G)$ is isometrically isomorphic to $E(G)/Z(G)$. In Section 3 we construct orthogonal basis vectors for the cycle and cut spaces and in Section 2.3 we compute their norms. They are used in Section 3 to construct a projection P_n from the edge space onto the cycle space of relatively small norm (Theorem 11). In Section 4 we show that P_n is close to being of minimal norm (Theorem 15). To prove this, we use the method of invariant projections as in Grünbaum [11], Rudin [27] and Andrew [2], and analyze projections that are invariant with respect to a certain group of isometries of the edge space.

Let X be a finite-dimensional normed space and let X_1 be any subspace of ℓ_∞ that is isometrically isomorphic to X. Recall that the *projection constant* of X, denoted $\lambda(X)$, is defined by

$$\lambda(X) = \inf\{\|P\| \colon P \colon \ell_\infty \to \ell_\infty \text{ is a projection with range } X_1\}.$$

(Note that $\lambda(X)$ is independent of the choice of X_1.)

In Section 5 we deduce from Theorems 11 and 15 reasonably sharp estimates of the projection constant of the space of Lipschitz functions on $\mathcal{L}_n$ (Theorem 16). We also present the results described above on the transportation cost space of $\mathcal{L}_n$. In Section 6 we sharpen the proof of Theorem B from [8] to obtain the exact projection constant of the space of Lipschitz functions on $D_{n,k}$.

In Section 7, for the convenience of the reader we give a direct proof in the finite case that if $\mathrm{TC}(X)$ is isometric to $\ell_1^{|X|-1}$ then X is a weighted tree and make a comment on the number of extreme points in the unit ball of $\mathrm{TC}(M)$.

Section 8 is devoted to simple examples of finite metric spaces, transportation cost spaces on which contain ℓ_∞^3 and ℓ_∞^4 isometrically. Earlier, more complicated finite spaces with this property were provided in [17]. It is an open question whether there exist a finite metric space M such that $\mathrm{TC}(M)$ contains ℓ_∞^5 isometrically.

2. Preliminaries

2.1. **Definitions and notation needed for the proofs.** Let us fix some notation for the Laakso graph $\mathcal{L}_n$. We denote the edge, cycle, and cut spaces of $\mathcal{L}_n$ by E_n, Z_n and C_n respectively. The usual ℓ_1,ℓ_2, and ℓ_∞ norms on E_n are denoted $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$. The usual inner product is denoted $\langle\cdot,\cdot\rangle$.

The edges of $\mathcal{L}_1$ are labelled as in Figure 3. We shall fix the reference orientation indicated by the arrows.

For the induction arguments which are used it will be convenient to label the 6 sub-$\mathcal{L}_{n-1}$'s of $\mathcal{L}_n$ as $A,\dots,F$ as shown in Figure 4. For $n \geq 2$, the edges of $\mathcal{L}_n$ inherit a reference orientation from $\mathcal{L}_1$ as indicated by the arrows in Figure 4. The edges of $\mathcal{L}_n$ are oriented from 'bottom' to 'top' in Figure 4.

For each $1 \leq j \leq n$, we shall use the term 'sub-$\mathcal{L}_j$' to refer to any of the copies of $\mathcal{L}_j$ contained in $\mathcal{L}_n$.

2.2. **The cycle and cut spaces of $\mathcal{L}_n$.** For each $1 \leq j \leq n$ and for each given sub-$\mathcal{L}_j$, Z_n contains the signed indicator function of the outer cycle (see Figure 3) contained in the given sub-$\mathcal{L}_j$. The collection of all such signed indicator functions is easily seen to be an algebraic basis of Z_n. Counting the total number of sub-$\mathcal{L}_j$'s , it follows that $\dim Z_n = (6^n-1)/5$, and hence $\dim C_n = (4\cdot 6^n + 1)/5$ since C_n is the orthogonal complement of Z_n. However, this basis of Z_n is difficult to work with because it is not orthogonal.

We shall now construct orthogonal bases for Z_n and C_n which will be used later to analyze projections onto Z_n.

$n = 1$: A vector in the edge space will be denoted by a vector

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix},$$

where x_i denotes the coefficient on the edge labelled i (see Figure 3).

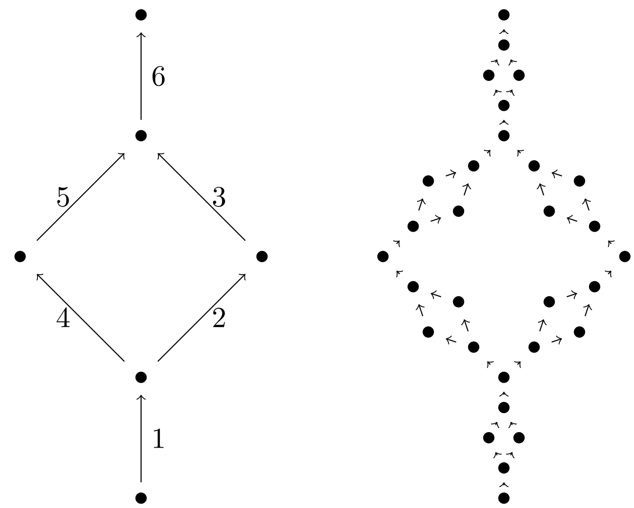

FIGURE 3. The Laakso graphs $\mathcal{L}_1$ and $\mathcal{L}_2$

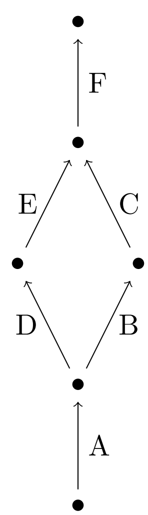

FIGURE 4. The Laakso graph $\mathcal{L}_n$

Note that $\dim Z_1 = 1$ and $\dim C_1 = 5$. It is easily seen that Z_1 is spanned by

$$h_1 = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}. \tag{4}$$

C_1, which is the orthogonal complement of Z_1, is easily seen to be spanned by the row vectors (which are orthogonal) of the following matrix:

$$\begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} \tag{5}$$

Note that these 6 vectors form an orthogonal basis of E_1.

$n = 2$: $\mathcal{L}_2$ is formed from $\mathcal{L}_1$ by replacing each edge of $\mathcal{L}_1$ by a copy of $\mathcal{L}_1$. Similarly, the edge vectors of $\mathcal{L}_2$ are obtained by replacing each coefficient x_i of an edge vector of $\mathcal{L}_1$ by the entries of a 6-dimensional vector.

In this way a vector in E_1 generates a vector in E_2 according to the following replacement rule: for each $x \in \mathbb{R}$,

$$x \mapsto \begin{bmatrix} x & x/2 & x/2 & x/2 & x/2 & x \end{bmatrix}.$$

We will describe this process of replacement as 'propagation'.

Define $f_1 \in C_1$ as follows:

$$f_1 = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

Note that

$$f_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

which expresses f_1 as the average of 2 indicator functions of paths connecting the bottom vertex of $\mathcal{L}_1$ to the top vertex. Hence h_1 propagates to an average of two signed indicator functions of cycles in $\mathcal{L}_2$. In particular, h_1 propagates to a vector h_2 in Z_2.

In addition to this vector, each of the 6 copies of $\mathcal{L}_1$ supports a 'new' cycle vector given by

$$\begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

(Its coefficients on the other five copies of $\mathcal{L}_1$ are all zero.) Note that this vector is orthogonal to the propagated vector since it is orthogonal to f_1.

The 5 basis vectors of C_1 propagate to form basis vectors of C_2. In addition, supported on each of the six copies of $\mathcal{L}_1$ we obtain 4 'new' orthogonal cut vectors given by the row vectors of the following matrix:

$$\begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Note that the row vectors are orthogonal to f_1. Hence the new cut vectors are orthogonal to the propagated cut vectors. The 5 propagated cut vectors and the 24 new cut vectors together form an orthogonal basis of the cut space C_2.

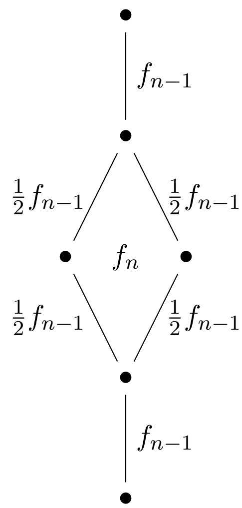

FIGURE 5. f_n defined on each copy of $\mathcal{L}_{n-1}$ in $\mathcal{L}_n$

$n \geq 3$: This is similar to the case $n = 2$. The orthogonal bases of Z_{n-1} and C_{n-1} propagate to collections of orthogonal vectors in Z_n and C_n. In addition, each of the 6^{n-1} copies of $\mathcal{L}_1$ supports one new cycle vector and 4 new cut vectors as above.

Let us check these claims. The claimed bases of Z_n and C_n are orthogonal and have the correct cardinality. So it suffices to check they they are contained in Z_n and C_n respectively. For $n \geq 2$, let h_n be the propagation of h_{n-1} and let f_n be the propagation of f_{n-1} (see Figure 5). It suffices to check that $h_n \in Z_n$. A straightfoward induction shows that f_n is the average of 2^n indicator functions of paths joining the bottom and top vertices of $\mathcal{L}_n$. Hence (see Figure 6) h_n is the average of 2^{n-1} signed indicator functions of large cycles in $\mathcal{L}_n$. In particular, $h_n \in Z_n$ as desired.

Recalling that C_n is the orthogonal complement of Z_n, the orthogonality of the basis guarantees that the claimed basis of C_n is indeed contained in C_n.

2.3. **Norms of cycle and cut vectors.** Note that

$$\|f_1\|_1 = 4, \|f_1\|_2^2 = 3.$$

For $n \geq 2$, define $f_n \in E_n$ inductively as shown in Figure 5. Note that

$$\|f_n\|_1 = 4\|f_{n-1}\|_1 = 4^n, \|f_n\|_2^2 = 3\|f_{n-1}\|_2^2 = 3^n.$$

Recall from (4) that $h_1 \in Z_1$ was defined by

$$h_1 = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \end{bmatrix}.$$

Now define $g_1 \in C_1$ by

$$g_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix},$$

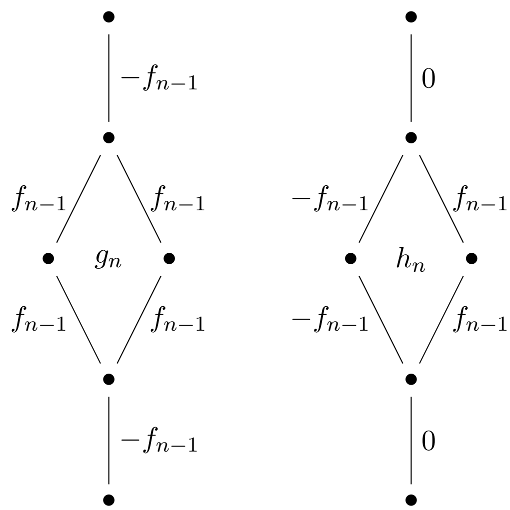

FIGURE 6. g_n and h_n defined on each copy of $\mathcal{L}_{n-1}$ in $\mathcal{L}_n$

and, for $n \geq 2$, define $g_n \in C_n$ and $h_n \in Z_n$ inductively as shown in Figure 6. Note that h_n is the cycle vector obtained from h_1 by repeated propagation, g_n is the cut vector obtained from g_1 by repeated propagation,

$$\|g_n\|_1 = 6\|f_{n-1}\|_1 = \frac{3}{2}4^n, \|g_n\|_2^2 = 6\|f_{n-1}\|_2^2 = 2 \cdot 3^n,$$

and

$$\|h_n\|_1 = 4\|f_{n-1}\|_1 = 4^n, \|h_n\|_2^2 = 4\|f_{n-1}\|_2^2 = \frac{4}{3} \cdot 3^n.$$

Hence, in particular,

$$\frac{\|g_n\|_1}{\|g_n\|_2^2} = \frac{\|h_n\|_1}{\|h_n\|_2^2} = (\frac{4}{3})^{n-1}. \tag{6}$$

Note that each sub-$\mathcal{L}_j$ supports a unique Z_n basis vector H_j of the form h_j and a unique C_n basis vector G_j of the form g_j. To justify this claim, let L_j be a sub-$\mathcal{L}_j$ of $\mathcal{L}_n$. For $j = 1$, G_1 and H_1 are the 'new' g_1 and h_1 basis vectors supported on L_1 arising in the passage from Z_{n-1} to Z_n and C_{n-1} to C_n described above. For $j > 1$, note that L_j evolves from a unique sub-$\mathcal{L}_1$ of $\mathcal{L}_{n-1-j}$, L_1' say. Let G_1' and H_1' be the g_1 and h_1 basis vectors supported on L_1'. Propagating G_1' and H_1' repeatedly $(j-1)$ times produces basis vectors G_j and H_j of the form g_j and h_j that are supported on L_j as claimed.

The next two lemmas will be used in Section 3.

Lemma 6. *Let $1 \leq j \leq n$ and let H_j and G_j be supported in some sub-$\mathcal{L}_j$, L_j, say. Then, for every edge vector e belonging to L_j, we have*

(1) $\langle e, H_j \rangle = 0 \Leftrightarrow \langle e, G_j \rangle < 0$.
(2) *If* $\langle e, H_j \rangle \neq 0$ *then* $\langle e, G_j \rangle > 0$ *and* $|\langle e, H_j \rangle| = \langle e, G_j \rangle$.

Proof. (1) From Figure 6, note that $\langle e, H_j \rangle = 0$ if and only if e belongs to the A or F sub-$\mathcal{L}_{j-1}$ of L_j if and only if $\langle e, G_j \rangle < 0$.

(2) If $\langle e, H_j \rangle \neq 0$ then e belongs to the B, C, D or E sub-$\mathcal{L}_{j-1}$. From Figure 6, note that $\langle e, G_j \rangle > 0$ and $|\langle e, H_j \rangle| = \langle e, G_j \rangle$. □

To state the next lemma, let us first fix some notation. For $1 \leq j \leq n$, let L_j be a sub-$\mathcal{L}_j$ of $\mathcal{L}_n$ such that $(L_j)_{j=1}^n$ is an increasing chain, i.e., $L_1 \subset L_2 \subset \cdots \subset L_n = \mathcal{L}_n$. Let S_j be the set of edge vectors contained in L_j, so that $(S_j)_{j=1}^n$ is also increasing. Finally, for $1 \leq j \leq n$, let G_j and H_j be the cut and cycle basis vectors corresponding to L_j (of the form g_j and h_j).

Lemma 7. *Let $1 \leq j \leq n$. Then for every $e \in S_1$, we have*

$$\langle e, G_j \rangle = (\frac{1}{2})^{\alpha_j} \operatorname{sgn}(\langle e, G_j \rangle) \quad \text{and} \quad \langle e, H_j \rangle = (\frac{1}{2})^{\alpha_j} \operatorname{sgn}(\langle e, H_j \rangle),$$

where $\alpha_1 = 0$ and, for $j \geq 2$, α_j is the cardinality of the set $\{1 \leq r < j \colon S_{r-1} \subset \operatorname{supp}(H_r)\}$ (here $\operatorname{sgn}(0) = 0$).

Proof. The result clearly holds for $j = 1$. So suppose that the result holds for $j = j_0$, where $1 \leq j_0 < n$. For $1 \leq j \leq n$, let F_j be the vector of the form f_j corresponding to L_j. From Figure 6, we have

$$|\langle e, G_{j_0+1} \rangle| = \langle e, F_{j_0} \rangle.$$

If $S_{j_0-1} \subset \operatorname{supp}(H_{j_0})$, then $\alpha_{j_0+1} = \alpha_{j_0} + 1$ and, from Figure 5,

$$\langle e, F_{j_0} \rangle = \frac{1}{2} \langle e, F_{j_0-1} \rangle = \frac{1}{2} |\langle e, G_{j_0} \rangle|,$$

(where $\langle e, F_0 \rangle = 1$ by convention in the case $j_0 = 1$). So by the inductive hypothesis,

$$\langle e, G_{j_0+1} \rangle = \frac{1}{2} |\langle e, G_{j_0} \rangle| \operatorname{sgn}(\langle e, G_{j_0+1} \rangle) = (\frac{1}{2})^{\alpha_{j_0+1}} \operatorname{sgn}(\langle e, G_{j_0+1} \rangle)$$

as desired. On the other hand, if S_{j_0-1} is disjoint from $\operatorname{supp}(H_{j_0})$, then $\alpha_{j_0+1} = \alpha_{j_0}$ and from Figure 5,

$$\langle e, F_{j_0} \rangle = \langle e, F_{j_0-1} \rangle = |\langle e, G_{j_0} \rangle|.$$

So by the inductive hypothesis,

$$\langle e, G_{j_0+1} \rangle = |\langle e, G_{j_0} \rangle| \operatorname{sgn}(\langle e, G_{j_0+1} \rangle) = (\frac{1}{2})^{\alpha_{j_0+1}} \operatorname{sgn}(\langle e, G_{j_0+1} \rangle)$$

as desired. The stated result for $\langle e, H_j \rangle$ follows from the result for $\langle e, G_j \rangle$ and Lemma 6. □

3. A projection onto the cycle space

In this section we define a projection P_n from E_n onto its cycle space Z_n which has relatively small (linear in n, i.e., logarithmic in $\dim(E_n)$) norm on $(E_n, \|\cdot\|_1)$.

Let us first observe that the *orthogonal* projection $\overline{P}_n$ of E_n onto Z_n has large (exponential in n) norm on $(E_n, \|\cdot\|_1)$.

Proposition 8.

$$\|\overline{P}_n\|_1 \geq (\frac{4}{3})^{n-1}.$$

Proof. Let e be the edge vector in Z_n corresponding to the 'lowest' edge (with respect to the 'bottom' to 'top' orientation) in the sub-$\mathcal{L}_{n-1}$ labelled as B. Then $\langle e, h_n\rangle = 1$ and $\langle e, h\rangle = 0$ if $h \neq h_n$ is any other basis vector of Z_n. Hence, using (6),

$$\|\overline{P}_n\|_1 \geq \|\overline{P}_n(e)\|_1 = \langle e, h_n\rangle \frac{\|h_n\|_1}{\|h_n\|_2^2} = (\frac{4}{3})^{n-1}.$$

□

The definition of P_n is inductive. P_1 is the orthogonal projection.

Suppose $n \geq 2$. We start the definition of P_n by setting $P_n(g_n) = 0$ and $P_n(g) = 0$ for every cut vector g in the orthogonal basis of C_n which is *not* of the form g_j for some sub-$\mathcal{L}_j$ $(1 \leq j \leq n-1)$. This is to be expected as we shall show in the next section that this holds for any projection which is invariant with respect to a natural group of isometries of E_n. Thus, to complete the definition, it suffices to define $P_n(g_j)$ for each sub-$\mathcal{L}_j$.

We shall label the six sub-$\mathcal{L}_{n-1}$'s as $A, \ldots, F$ as shown in Figure 4. On A and F we define P_n to be a copy of P_{n-1}. So it suffices to define $P_n(g_j)$ for all g_j supported on a a sub-$\mathcal{L}_j$ contained in B, C, D or E. The definition of $P_n(g_j)$ will proceed *backwards* from $j = n-1$ to $j = 1$.

Let S_{n-1} be the set of edge vectors of any one of B, C, D or E. Now let S_{n-2} be the set of edge vectors of any one of the 6 sub-$\mathcal{L}_{n-2}$'s supported in S_{n-1}. Continue in this way to obtain a chain $S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1$. Finally, let e be one of the 6 edge vectors contained in S_1. Note that S_1 uniquely determines the chain $(S_j)_{j=1}^{n-1}$ and that every edge vector e in the support of B, C, D, or E determines a unique choice of S_1.

For each $1 \leq j \leq n-1$, let G_j denote the g_j cut vector and let H_j denote the h_j cycle vector corresponding to the sub-$\mathcal{L}_j$ supported on S_j. We shall define $P_n(G_j)$ inductively along the chain $(S_j)_{j=1}^{n-1}$ starting with $j = n-1$. By varying the chain we define $P_n(G_j)$ for every cut vector in the orthogonal basis of C_n which is of the form G_j for some sub-$\mathcal{L}_j$ $(1 \leq j \leq n-1)$. Since each sub-$\mathcal{L}_j$ occurs in several different chains, we must also check that $P_n(G_j)$ is well-defined.

The motivating idea behind this definition is a 'balancing' of certain norms which is described in (iv) below. However, since the proof is lengthy and not particularly intuitive, we will describe the strategy before going into the details. The definition of $P_n(G_j)$ will involve a sequence of vectors $(X_j)_{j=1}^n$ and sequences of scalars $(x_j)_{j=1}^n$ and $(a_j)_{j=1}^n$, which are defined inductively. The strategy behind the definition of P_n and the proof of Theorem 11 below is as follows:

(i) X_j is completely determined by S_{j-1} and is defined inductively as a linear combination of $H_j, H_{j+1}, \ldots, h_n$.
(ii) The definition of X_j given by (10) has two cases, depending on whether or not S_{j-1} is contained in the support of H_j (equivalently, whether or not $e \in \operatorname{supp}(H_j)$).
(iii) The choice of a_j as defined by (9) ensures that X_j has roughly the same $\|\cdot\|_1$ norm in both cases.
(iv) Hence $P_n(G_j)$, as defined by (8), has roughly the same norm in both cases of the definition of X_{j+1}. It is this balancing which ultimately leads to a projection of relatively small norm. (Note also that $P_n(G_j)$ is a certain linear combination of $H_{j+1}, H_{j+2}, \ldots, h_n$.)
(v) The choice of a_j ensures that $\|X_j\|_1 \leq x_j := (1-a_j)x_{j+1}$.
(vi) It is shown in Lemma 9 that $X_1 = P_n(e)$, and hence $\|P_n(e)\|_1 \leq x_1$. This is the key estimate in the proof of Theorem 11.
(vii) $(x_j)_{j=1}^n$ satisfies a recurrence relation which is solved in Lemma 10. This leads to the estimate $\|P_n\|_1 \leq (n+1)/2$, which is proved in Theorem 11.

Let us now go through the details of the definition of $P_n(G_j)$ starting with $j = n-1$. Set

$$X_n = \operatorname{sgn}(\langle e, h_n\rangle)\frac{h_n}{\|h_n\|_2^2} \quad \text{and} \quad x_n = \|X_n\|_1 = (\frac{4}{3})^{n-1}, \tag{7}$$

where $\operatorname{sgn}(a)$ is the sign of a. Define

$$P_n(\frac{G_{n-1}}{\|G_{n-1}\|_2^2}) = a_{n-1}X_n,$$

where a_{n-1} is defined by the equation

$$(1-a_{n-1})(\frac{4}{3})^{n-1} = (\frac{1}{2}+a_{n-1})(\frac{4}{3})^{n-1} + (\frac{4}{3})^{n-2}.$$

(Note that, in fact, $a_{n-1} = -1/8$.) Now set

$$X_{n-1} = \begin{cases} (\frac{1}{2}+a_{n-1})X_n + \operatorname{sgn}(\langle e, H_{n-1}\rangle)\frac{H_{n-1}}{\|H_{n-1}\|_2^2}, & e \in \operatorname{supp}(H_{n-1}), \\ (1-a_{n-1})X_n, & e \notin \operatorname{supp}(H_{n-1}). \end{cases}$$

Since $1-a_{n-1}=9/8>0$ and $\dfrac{1}{2}+a_{n-1}=3/8>0$, the triangle inequality and (6) give

$$\begin{aligned}\|X_{n-1}\|_1 &\le [(\frac{1}{2}+a_{n-1})\|X_n\|_1+\frac{\|H_{n-1}\|_1}{\|H_{n-1}\|_2^2}]\vee(1-a_{n-1})\|X_n\|_1\\ &=[(\frac{1}{2}+a_{n-1})(\frac{4}{3})^{n-1}+(\frac{4}{3})^{n-2}]\vee(1-a_{n-1})(\frac{4}{3})^{n-1}\\ &=(1-a_{n-1})(\frac{4}{3})^{n-1}\\ &=(1-a_{n-1})\|X_n\|_1.\end{aligned}$$

Set $x_{n-1}=(1-a_{n-1})\|X_n\|_1$. Then $\|X_{n-1}\|_1\le x_{n-1}$.

Let us now turn to the inductive step, which is similar to the case $j=n-1$. Suppose that $1\le j<n-1$ and that X_{j+1}, x_{j+1}, and $P_n(G_{j+1})$ have been defined with $\|X_{j+1}\|_1\le x_{j+1}$. Now define

$$P_n(\frac{G_j}{\|G_j\|_2^2})=a_jX_{j+1}, \tag{8}$$

where a_j is defined by the equation

$$(1-a_j)x_{j+1}=(\frac{1}{2}+a_j)x_{j+1}+(\frac{4}{3})^{j-1}. \tag{9}$$

Set

$$X_j=\begin{cases}(\frac{1}{2}+a_j)X_{j+1}+\operatorname{sgn}(\langle e,H_j\rangle)\frac{H_j}{\|H_j\|_2^2}, & e\in\operatorname{supp}(H_j),\\ (1-a_j)X_{j+1}, & e\notin\operatorname{supp}(H_j).\end{cases} \tag{10}$$

It is worth observing that, for $j\ge 2$, X_j does not depend on the particular choice of e from S_1. Hence, for $j\ge 1$, $P_n(G_j)$ defined by (8) is also independent of the choice of e as required. But we prove below (Lemma 9) that $X_1=P_n(e)$, which does depend on the choice of e.

We prove in Lemma 10 below that $\dfrac{1}{2}+a_j>0$ and $1-a_j>0$. Hence, by the triangle inequality and (6),

$$\begin{aligned}\|X_j\|_1 &\le [(\frac{1}{2}+a_j)\|X_{j+1}\|_1+\frac{\|H_j\|_1}{\|H_j\|_2^2}]\vee(1-a_j)\|X_{j+1}\|_1\\ &\le [(\frac{1}{2}+a_j)x_{j+1}+(\frac{4}{3})^{j-1}]\vee(1-a_j)x_{j+1}\\ &=(1-a_j)x_{j+1}.\end{aligned}$$

Finally, set $x_j=(1-a_j)x_{j+1}$ to complete the inductive step.

To check that $P_n(G_j)$ as given by (8) is well-defined, we need to check that it depends only on $\operatorname{supp}(G_j)=S_j$. To see this, note that S_j determines its 'ancestors' $S_{j+1},\dots,S_{n-1}$ uniquely. Moreover, the definition of X_{j+1} (see (10) and replace j by $j+1$) actually depends only on S_j since $\operatorname{sgn}(\langle e,H_{j+1}\rangle)$ is simply the (constant) sign of H_{j+1} on S_j. Hence $P_n(G_j)$ is indeed well-defined.

By considering every chain $S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1$, we define $S(g)$ for every cut vector g of the form g_j for some sub-$\mathcal{L}_j$.

The definition of P_n is now complete. (Recall that we started the definition by setting $P_n(g_n) = 0$ and $P_n(g) = 0$ for all other cut vectors g in the orthogonal basis of C_n described above.)

Lemma 9. $P_n(e) = X_1$.

Proof. Using Lemma 6 and the fact (see (8)) that

$$P_n(\frac{G_j}{\|G_j\|_2^2}) = a_j X_{j+1},$$

we can combine the two cases in the definition (10) of X_j as follows:

$$X_j = (\frac{1}{2})^{\varepsilon_j} X_{j+1} + \operatorname{sgn}(\langle e, G_j\rangle)\frac{P_n(G_j)}{\|G_j\|_2^2} + \operatorname{sgn}(\langle e, H_j\rangle)\frac{H_j}{\|H_j\|_2^2},$$

where

$$\varepsilon_j = \begin{cases} 1, & S_{j-1} \subset \operatorname{supp}(H_j), \\ 0, & S_{j-1} \cap \operatorname{supp}(H_j) = \emptyset \end{cases}$$

and setting $\operatorname{sgn}(0) = 0$. After repeated application of this formula, starting at $j = 1$ and ending at $j = n-1$, and then substituting (see (7))

$$X_n = \operatorname{sgn}(\langle e, h_n\rangle)\frac{h_n}{\|h_n\|_2^2},$$

we obtain

$$X_1 = \sum_{j=1}^{n-1}(\frac{1}{2})^{\alpha_j}[\operatorname{sgn}(\langle e, G_j\rangle)\frac{P_n(G_j)}{\|G_j\|_2^2} + \operatorname{sgn}(\langle e, H_j\rangle)\frac{H_j}{\|H_j\|_2^2}] + (\frac{1}{2})^{\alpha_n}\operatorname{sgn}(\langle e, h_n\rangle)\frac{h_n}{\|h_n\|_2^2},$$

where $\alpha_1 = 0$ and, for $j \geq 2$, α_j is the cardinality of the set $\{1 \leq r < j \colon S_{r-1} \subset \operatorname{supp}(H_r)\}$. By Lemma 7, for $1 \leq j \leq n-1$,

$$\langle e, G_j\rangle = (\frac{1}{2})^{\alpha_j}\operatorname{sgn}(\langle e, G_j\rangle)$$

and

$$\langle e, H_j\rangle = (\frac{1}{2})^{\alpha_j}\operatorname{sgn}(\langle e, H_j\rangle)$$

and

$$\langle e, h_n\rangle = (\frac{1}{2})^{\alpha_n}\operatorname{sgn}(\langle e, h_n\rangle).$$

Hence

$$\begin{aligned} X_1 &= \sum_{j=1}^{n-1}[\langle e, G_j\rangle\frac{P_n(G_j)}{\|G_j\|_2^2} + \langle e, H_j\rangle\frac{H_j}{\|H_j\|_2^2}] + \langle e, h_n\rangle\frac{h_n}{\|h_n\|_2^2} \\ &= P_n([\sum_{j=1}^{n-1}\langle e, \frac{G_j}{\|G_j\|_2}\rangle\frac{G_j}{\|G_j\|_2} + \langle e, \frac{H_j}{\|H_j\|_2}\rangle\frac{H_j}{\|H_j\|_2}] + \langle e, \frac{h_n}{\|h_n\|_2}\rangle\frac{h_n}{\|h_n\|_2}) \\ &= P_n(e). \end{aligned}$$

To see the last line of the above, note that if $\langle e, g\rangle \neq 0$, for g belonging to the othogonal basis of C_n, then either $g = G_j$ for some $1 \le j \le n-1$ or $P_n(g) = 0$. This is because we began the definition of P_n by setting $P_n(g_n) = 0$ and $P_n(g) = 0$ for every cut vector g in the orthogonal basis of C_n which is *not* of the form g_j for some sub-$\mathcal{L}_j$. On the other hand, if g is of the form g_j and $\langle e, g\rangle \neq 0$ then $\text{supp}(g) = S_j$, i.e., $g = G_j$. Similarly, if $\langle e, h\rangle \neq 0$, for h belonging to the orthogonal basis of Z_n, then either $h = H_j$ or $h = h_n$. So the above expression for X_1 is simply P_n applied to the expansion of e with respect to the othogonal basis of E_n.

□

Lemma 10. $x_1 = \dfrac{n+1}{2}$ *and* $\min(1-a_j, \frac{1}{2}+a_j) > 0$ *for* $1 \le j < n-1$.

Proof. Recall that $x_n = (4/3)^{n-1}$ (see (7)) and that, for $1 \le j \le n-1$, x_j satisfies the recurrence

$$x_j = (1-a_j)x_{j+1} = (\frac{1}{2}+a_j)x_{j+1} + (\frac{4}{3})^{j-1}$$

which serves to define a_j for $1 \le j \le n-1$. Hence

$$\begin{aligned} x_j &= \frac{1}{2}[(1-a_j)x_{j+1} + (\frac{1}{2}+a_j)x_{j+1} + (\frac{4}{3})^{j-1}] \\ &= \frac{3}{4}x_{j+1} + \frac{1}{2}(\frac{4}{3})^{j-1}. \end{aligned}$$

The solution to this recurrence is

$$x_j = \frac{n+2-j}{2}(\frac{4}{3})^{j-1}.$$

Note that

$$a_j x_{j+1} = x_{j+1} - x_j = (\frac{4}{3})^j[-\frac{1}{4} + \frac{n-j}{8}].$$

Hence $a_{n-1} = -\dfrac{1}{8}, a_{n-2} = 0$, and $0 < a_j < 1$ for $1 \le j \le n-3$. In all cases $\min(1-a_j, \dfrac{1}{2}+a_j) > 0$. □

Theorem 11. $\|P_n\|_1 \le \dfrac{n+1}{2}$.

Proof. Recall that P_1 is the orthogonal projection onto Z_1:

$$P_1(e_i) = \begin{cases} \frac{\pm 1}{4}(e_2+e_3-e_4-e_5), & i = 2,3,4,5 \\ 0, & i = 1,6. \end{cases}$$

Clearly, $\|P_1\|_1 = 1$. Now suppose $n \ge 2$. If e is an edge vector belonging to the A or F sub-$\mathcal{L}_{n-1}$, then, by the inductive hypothesis,

$$\|P_n(e)\|_1 \le \|P_{n-1}\|_1 \le \frac{n}{2}.$$

On the other hand, if e belongs to the B, C, D or E sub-$\mathcal{L}_{n-1}$, then $P_n(e) = X_1$ for the chain $(S_j)_{j=1}^{n-1}$ with $e \in S_1$, so by Lemma 10,

$$\|P_n(e)\|_1 = \|X_1\|_1 \leq x_1 = \frac{n+1}{2}.$$

Hence

$$\|P_n\|_1 = \max_e \|P_n(e)\|_1 \leq \frac{n+1}{2}.$$

□

4. Invariant Projections

In this section we prove that the projection P_n constructed in the previous section is close to being optimal. First we show that we may restrict attention to projections that are 'invariant' with respect to a certain group of isometries of E_n. Then we show that P_n is close to being optimal in the sense that its operator norm is of the same order.

First, let us define a group of isometries of $(E_n, \|\cdot\|_2)$. To that end, let us say that a cut vector g belonging to the orthogonal basis of the cut space C_n is **special** if g is of the the form g_j for some sub-$\mathcal{L}_j$ for $1 \leq j \leq n$. We shall say that g is **non-special** if g is not of the form g_j and g is not the unique cut vector propagated by $\begin{bmatrix}1 & 1/2 & 1/2 & 1/2 & 1/2 & 1\end{bmatrix}$.

If g is a non-special cut vector then there will be a smallest sub-$\mathcal{L}_j$ ($1 \leq j \leq n$) which contains its support. Let us call this the *support sub-$\mathcal{L}_j$* of g. Let ψ_g be the natural isometry of E_n induced by interchanging $\{g > 0\}$ and $\{g < 0\}$. Since there are three types of non-special vector, namely those cut vectors propagated by the second, third, and fourth rows of (5), ψ_g is effectuated by either (a) interchanging the B and C sub-$\mathcal{L}_{j-1}$ of its support (using the inductively defined isomorphism between B and C and $\mathcal{L}_{j-1}$), or (b) interchanging the D and E sub-$\mathcal{L}_{j-1}$, or (c) interchanging the A and F sub-$\mathcal{L}_{j-1}$. Note that Z_n and C_n are ψ_g-invariant subspaces.

Similarly, each Z_n basis vector h has a support sub-$\mathcal{L}_j$. Let ϕ_h be the natural isometry induced by interchanging $\{h > 0\}$ and $\{h < 0\}$. Then ϕ_h is effectuated by interchanging the B and E sub-$\mathcal{L}_{j-1}$ and the C and D sub-$\mathcal{L}_{j-1}$ of the support sub-$\mathcal{L}_j$ of h. Note that Z_n and C_n are ϕ_h-invariant subspaces.

Note that $\phi_h^* = \phi_h = \phi^{-1}$ and $\psi_g^* = \psi_g = \psi_g^{-1}$ when considered as isometries of the Euclidean space $(E_n, \|\cdot\|_2)$.

Let G be the (finite) group generated by the collection of all ψ_g and ϕ_h isometries. Let Q be any projection form E_n onto Z_n. Then

$$P = \frac{1}{|G|} \sum_{\theta \in G} \theta^{-1} Q \theta$$

satisfies $\|P\|_1 \leq \|Q\|_1$, and $P\theta = \theta P$ for all $\theta \in G$. Moreover, P is also a projection onto Z_n since Z_n and C_n are θ-invariant for each $\theta \in G$.

Lemma 12. *If g is non-special or if g is the (unique) cut vector propagated by $\begin{bmatrix}1 & 1/2 & 1/2 & 1/2 & 1/2 & 1\end{bmatrix}$ then $P(g) = 0$.*

Proof. Since $P(g) \in Z_n$ it suffices to show that $\langle P(g), h\rangle = 0$ for every h belonging to the basis of Z_n. If $\operatorname{supp}(g) \subseteq \operatorname{supp}(h)$ then $\psi_g(g) = -g$ and $\psi_g(h) = h$. So

$$\langle P(g), h\rangle = \langle P(g), \psi_g(h)\rangle = \langle \psi_g(P(g)), h\rangle = \langle P(\psi_g(g)), h\rangle = -\langle P(g), h\rangle.$$

On the other hand, if $\operatorname{supp}(h) \subseteq \operatorname{supp}(g)$ or $\operatorname{supp}(h) \cap \operatorname{supp}(g) = \emptyset$ then $\phi_h(h) = -h$ and $\phi_h(g) = g$. So

$$\langle P(g), h\rangle = \langle P(\phi_h(g)), h\rangle = \langle \phi_h(P(g)), h\rangle = \langle P(g), \phi_h(h)\rangle = -\langle P(g), h\rangle.$$

Hence, in both cases, $\langle P(g), h\rangle = 0$. □

Lemma 13. *If g is a special cut vector then*

$$P(g) \in \operatorname{span}\{h\colon \operatorname{supp}(g) \subset \operatorname{supp}(h)\}.$$

In particular, $P(g_n) = 0$.

Proof. If $\operatorname{supp}(h) \subseteq \operatorname{supp}(g)$ or $\operatorname{supp}(h) \cap \operatorname{supp}(g) = \emptyset$, then, as above, $\langle P(g), h\rangle = 0$, which gives the result. □

The following lemma will be needed in the proof of Theorem 15 below.

Lemma 14. *Let $(H_j)_{j=1}^n$ be a chain of cycle vectors such that H_j is of type h_j and $\operatorname{supp}(H_j) \subset \operatorname{supp}(H_{j+1})$ for each $1 \le j < n$. Then*

$$\|\sum_{j=1}^n a_j H_j\|_1 \ge \frac{3}{4}\sum_{j=1}^n |a_j|\|H_j\|_1$$

for all scalars $(a_j)_{j=1}^n$.

Proof. Note that, for each $2 \le j \le n$,

$$\|H_j|_{\operatorname{supp}(H_{j-1})}\|_1 = \frac{1}{8}\|H_j\|_1.$$

Hence

$$\begin{aligned}\|\sum_{j=1}^n a_j H_j\|_1 &= |a_n|\|H_n|_{\operatorname{supp}(H_n)\setminus \operatorname{supp}(H_{n-1})}\|_1 + \|\sum_{j=1}^{n-1} a_j H_j + a_n H_n|_{\operatorname{supp}(H_{n-1}}\|_1\\ &\ge |a_n|\|H_n\|_1 + \|\sum_{j=1}^{n-1} a_j H_j\|_1 - 2|a_n|\|H_n|_{\operatorname{supp}(H_{n-1})}\|_1\\ &= \frac{3}{4}|a_n|\|H_n\|_1 + \|\sum_{j=1}^{n-1} a_j H_j\|_1.\end{aligned}$$

Iterating this calculation yields the result. □

Theorem 15. *Let Q be any projection from E_n onto Z_n. Then $\|Q\|_1 \ge \frac{3}{8}(n+1)$.*

Proof. Let P be the invariant projection associated to Q. We shall prove that $\|P\|_1 \geq \frac{3}{8}(n+1)$, which implies the result since $\|P\|_1 \leq \|Q\|_1$.

The analysis of P is very similar to the analysis of P_n in the previous section. In particular, we will define an auxiliary sequence of vectors $(X_j)_{j=1}^n$ and an auxiliary sequence of scalars $(x_j)_{j=1}^n$. The goal is to *construct* a chain $(S_j)_{j=0}^{n-1}$, with $S_0 = \{e\}$, such that $\|P(e)\|_1$ is large, i.e., comparable to $\|P_n\|_1$. This is a chain which (roughly speaking) maximizes $\|X_j\|_1$ at each bifurcation.

To that end, we shall inductively define a chain of cycle vectors $(H_j)_{j=1}^n$ such that H_j is of type h_j and $\mathrm{supp}(H_j) \subset \mathrm{supp}(H_{j+1})$ for each $1 \leq j < n$. To start the induction, set $H_n = h_n$. To simplify the calculation of the norm we define an equivalent norm $\|\!|\cdot|\!\|$ on $\mathrm{span}(H_j)_{j=1}^n$ which is easier to work with:

$$\Big|\!\Big|\!\Big|\sum_{j=1}^n a_j H_j\Big|\!\Big|\!\Big| = \sum_{j=1}^n |a_j| \|H_j\|_1.$$

By Lemma 14

$$\Big\|\sum_{j=1}^n a_j H_j\Big\|_1 \leq \Big|\!\Big|\!\Big|\sum_{j=1}^n a_j H_j\Big|\!\Big|\!\Big| \leq \frac{4}{3}\Big\|\sum_{j=1}^n a_j H_j\Big\|_1.$$

Inductively, we define vectors $(X_j)_{j=1}^n$ and a decreasing chain $S_{n-1} \supset S_{n-2} \supset \cdots \supset S_1$ such that S_j is the support of a sub-$\mathcal{L}_j$. To start the inductive definition, set

$$X_n = \frac{H_n}{\|H_n\|_2^2} \quad \text{and} \quad x_n = |\!|\!|X_n|\!|\!| = \|X_n\|_1 = (\frac{4}{3})^{n-1}$$

and let $S_{n-1} \subset \{h_n > 0\}$. Set

$$x_{n-1} = \Big|\!\Big|\!\Big|X_n - P(\frac{G_{n-1}}{\|G_{n-1})\|_2^2})\Big|\!\Big|\!\Big| \vee \Big|\!\Big|\!\Big|\frac{X_n}{2} + P(\frac{G_{n-1}}{\|G_{n-1}\|_2^2}) + \frac{H_{n-1}}{\|H_{n-1}\|_2^2}\Big|\!\Big|\!\Big|.$$

Averaging the two vectors above and using convexity of $|\!|\!|\cdot|\!|\!|$,

$$\begin{aligned} x_{n-1} &\geq \Big|\!\Big|\!\Big|\frac{3}{4}X_n + \frac{1}{2}\frac{H_{n-1}}{\|H_{n-1}\|_2^2}\Big|\!\Big|\!\Big| \\ &= \frac{3}{4}|\!|\!|X_n|\!|\!| + \frac{1}{2}\Big\|\frac{H_{n-1}}{\|H_{n-1}\|_2^2}\Big\|_1 \\ &= \frac{3}{4}x_n + \frac{1}{2}(\frac{4}{3})^{n-2} \end{aligned}$$

by (6). If

$$x_{n-1} = \Big|\!\Big|\!\Big|\frac{X_n}{2} + P(\frac{G_{n-1}}{\|G_{n-1}\|_2^2}) + \frac{H_{n-1}}{\|H_{n-1}\|_2^2}\Big|\!\Big|\!\Big|,$$

set

$$X_{n-1} = \frac{X_n}{2} + P(\frac{G_{n-1}}{\|G_{n-1}\|_2^2}) + \frac{H_{n-1}}{\|H_{n-1}\|_2^2}$$

and choose $S_{n-2} \subset \{H_{n-1} > 0\}$. Otherwise, set

$$X_{n-1} = X_n - P(\frac{G_{n-1}}{\|G_{n-1}\|_2^2})$$

and choose $S_{n-2} \subset S_{n-1}$ disjoint from $\mathrm{supp}(H_{n-1})$.

We now describe the inductive step which is similar to the case $j = n-1$. Suppose that $1 \le j < n-1$ and that S_i, X_i and x_i have been defined for $i = j+1, \dots, n$ with $S_j \subset S_{j+1} \subset \cdots \subset S_n$ and with $X_i \in \mathrm{span}\{H_k \colon i \le k \le n\}$. Set $x_i = |||X_i|||$

Let G_j and H_j be the cut and cycle vectors whose support sub-$\mathcal{L}_j$ is S_j. Note that, by Lemma 13,

$$P(G_j) \in \mathrm{span}\{H_i \colon j+1 \le i \le n\}$$

and hence

$$x_j = |||X_{j+1} - P(\frac{G_j}{\|G_j\|_2^2})||| \vee |||\frac{X_{j+1}}{2} + P(\frac{G_j}{\|G_j\|_2^2}) + \frac{H_j}{\|H_j\|_2^2}|||$$

is well-defined. Moreover, by convexity,

$$\begin{aligned} x_j &\ge |||\frac{3}{4}X_{j+1} + \frac{1}{2}\frac{H_j}{\|H_j\|_2^2}||| \\ &= \frac{3}{4}|||X_{j+1}||| + \frac{1}{2}\frac{\|H_j\|_1}{\|H_j\|_2^2} \end{aligned}$$

(since $X_{j+1} \in \mathrm{span}\{H_k \colon \, \colon j+1 \le k \le n\}$)

$$= \frac{3}{4}x_{j+1} + \frac{1}{2}(\frac{4}{3})^{j-1}.$$

If

$$x_j = |||\frac{X_{j+1}}{2} + P(\frac{G_j}{\|G_j\|_2^2}) + \frac{H_j}{\|H_j\|_2^2}|||,$$

set

$$X_j = \frac{X_{j+1}}{2} + P(\frac{G_j}{\|G_j\|_2^2}) + \frac{H_j}{\|H_j\|_2^2}$$

and choose $S_{j-1} \subset \{H_j > 0\}$. Otherwise, set

$$X_j = X_{j+1} - P(\frac{G_j}{\|G_j)\|_2^2})$$

and choose $S_{j-1} \subset S_j$ disjoint from $\mathrm{supp}(H_j)$. Note that in both cases we have $X_j \in \mathrm{span}\{H_k \colon j \le k \le n\}$ as required. This completes the inductive definition. Note that $S_0 = \{e\}$ for some edge vector e. Moreover, using Lemma 6 we can combine both cases to obtain, for $1 \le j \le n-1$,

$$X_j = (\frac{1}{2})^{\varepsilon_j} X_{j+1} + \mathrm{sgn}(\langle e, G_j \rangle)\frac{P_n(G_j)}{\|G_j\|_2^2} + \mathrm{sgn}(\langle e, H_j \rangle)\frac{H_j}{\|H_j\|_2^2},$$

where

$$\varepsilon_j = \begin{cases} 1, & S_{j-1} \subset \mathrm{supp}(H_j), \\ 0, & S_{j-1} \cap \mathrm{supp}(H_j) = \emptyset. \end{cases}$$

Arguing as in the proof of Lemma 9 it follows that

$$\begin{aligned}X_1 &= \sum_{j=1}^{n-1}[\langle e, G_j\rangle \frac{P(G_j)}{\|G_j\|_2^2} + \langle e, H_j\rangle \frac{H_j}{\|H_j\|_2^2}] + \langle e, H_n\rangle \frac{H_n}{\|H_n\|_2^2} \\ &= P(\sum_{j=1}^{n-1}[\langle e, \frac{G_j}{\|G_j\|_2}\rangle \frac{G_j}{\|G_j\|_2} + \langle e, \frac{H_j}{\|H_j\|_2}\rangle \frac{H_j}{\|H_j\|_2}] + \langle e, \frac{H_n}{\|H_n\|_2}\rangle \frac{H_n}{\|H_n\|_2}) \\ &= P(e)\end{aligned}$$

To see this, note that $P(g_n) = 0$ by Lemma 13 and $P(g) = 0$ by Lemma 12 unless g is a special cut vector of the form g_j for some sub-$\mathcal{L}_j$. Note also that if h is of the form h_j and g is of the form g_j for some sub-$\mathcal{L}_j$, then $\langle e, h\rangle \neq 0$ only if $h = H_j$ $(1 \le j \le n)$ and $\langle e, g\rangle \neq 0$ only if $g = G_j$ $(1 \le j \le n)$. So the above expression for X_1 is simply P applied to the expansion of e with respect to the othogonal basis of E_n.

Finally,

$$\|P\|_1 \ge \|P(e)\|_1 = \|X_1\|_1 \ge \frac{3}{4}\|X_1\| = \frac{3}{4}x_1 \ge \frac{3}{4}(\frac{n+1}{2}).$$

The last inequality follows from the solution of the recurrence in Lemma 10 since

$$x_j \ge \frac{3}{4}x_{j+1} + \frac{1}{2}(\frac{4}{3})^{j-1}, x_n = (\frac{4}{3})^{n-1}.$$

□

5. Applications to the transportation cost space of $\mathcal{L}_n$

Theorem 16. *The projection constant of* $\mathrm{Lip}_0(\mathcal{L}_n)$ *satisfies*

$$\frac{3n-5}{8} \le \lambda(\mathrm{Lip}_0(\mathcal{L}_n)) \le \frac{n+3}{2}.$$

Proof. Note that $\mathrm{Lip}_0(\mathcal{L}_n) = (\mathrm{TC}(\mathcal{L}_n))^*$ is isometrically isomorphic to $(C_n, \|\cdot\|_\infty) \subset (E_n, \|\cdot\|_\infty)$ by (3), since $C_n = Z_n^\perp$. Let P_n be the projection from $(E_n, \|\cdot\|_1)$ onto Z_n constructed in Section 3. Then $I - P_n^*$ is a projection from $(E_n, \|\cdot\|_\infty)$ onto $Z_n^\perp = C_n$. Thus,

$$\lambda(\mathrm{Lip}_0(\mathcal{L}_n)) \le \|I - P_n^*\| \le 1 + \|P_n\| \le 1 + \frac{n+1}{2} = \frac{n+3}{2}.$$

Now suppose Q is any projection from $(E_n, \|\cdot\|_\infty)$ onto C_n. Then $I - Q^*$ is a projection from $(E_n, \|\cdot\|_1)$ onto Z_n. So, by Theorem 15,

$$\|Q\| \ge \|I - Q^*\| - 1 \ge \frac{3}{8}(n+1) - 1 = \frac{3n-5}{8}.$$

So $\lambda(\mathrm{Lip}_0(\mathcal{L}_n)) \ge (3n-5)/8$. □

Corollary 17. *The Banach-Mazur distance from* $\mathrm{TC}(\mathcal{L}_n)$ *to* ℓ_1^N*, where* $N(n) = (4 \cdot 6^n + 1)/5$ *is the dimension of* $\mathrm{TC}(\mathcal{L}_n)$*, satisfies*

$$d_{BM}(\mathrm{TC}(\mathcal{L}_n), \ell_1^N) \ge (3n-5)/8.$$

Proof. By duality,

$$d_{BM}(\mathrm{TC}(\mathcal{L}_n), \ell_1^N) = d_{BM}(\mathrm{Lip}_0(\mathcal{L}_n), \ell_\infty^N) \geq \lambda(\mathrm{Lip}_0(\mathcal{L}_n)) \geq \frac{3n-5}{8}.$$

□

Remark 18. The interpretation of this corollary in terms of transportation costs is as follows. For each $1 \leq j \leq N$, let x_j be any transportation plan on $\mathcal{L}_n$ of unit cost. Then there exists an absolutely convex combination $\sum_{j=1}^N a_j x_j$ ($\sum_{j=1}^N |a_j| = 1$) such that

$$\|\sum_{j=1}^N a_j x_j\|_{\mathrm{TC}} \leq \frac{8}{3n-5} \qquad (n \geq 2).$$

In contrast to the diamond graphs D_n [8, Theorem 6.5], we have not been able to prove a good upper bound for the Banach-Mazur distance from $\mathrm{TC}(\mathcal{L}_n)$ to ℓ_1^N. However, we have the following matching upper bound for a linear embedding of $\mathrm{TC}(\mathcal{L}_n)$ into ℓ_1.

Corollary 19. *There exists $X_n \subset (E_n, \|\cdot\|_1)$ such that $d_{BM}(\mathrm{TC}(\mathcal{L}_n), X_n) \leq (n+3)/2$.*

Proof. Let P_n be the projection constructed in Section 3. Then, setting $X_n = \ker P_n$, Theorem 11 yields

$$d_{BM}(\mathrm{TC}(\mathcal{L}_n), X_n) = d_{BM}((E_n/Z_n, \|\cdot\|_1), X_n) \leq \|I - P_n\|_1 \leq \frac{n+3}{2}. \quad \square$$

Remark 20. Actually, as we remarked in the Introduction, for ever finite metric space X, $\mathrm{TC}(X)$ admits a linear embedding into $L_1[0,1]$ with distortion $\leq C \ln |X|$, see [5, 9, 13]. Corollary 19 is just a slightly more precise statement of this fact for $\mathrm{TC}(\mathcal{L}_n)$.

For the diamond graph D_n, the transportation cost space $\mathrm{TC}(D_n)$ has a natural monotone Schauder basis which leads to a matching upper bound for the Banach-Mazur distance. The difficulty in obtaining the same result for $\mathrm{TC}(\mathcal{L}_n)$ stems from the fact that the orthogonal basis of C_n constructed above is *not* a Schauder basis in the $\mathrm{TC}(\mathcal{L}_n)$ norm. In fact, the collection of special cut vectors g_j in $(C_n, \|\cdot\|_{\mathrm{TC}})$ does not admit a bounded biorthogonal system (uniformly in n).

To make this precise, for each $1 \leq j \leq n-1$, let g_j^i ($1 \leq i \leq 6^{n-j}$) be an enumeration of the 6^{n-j} basis vectors supported on a sub-$\mathcal{L}_j$. Note that $\mathrm{TC}(\mathcal{L}_n)$ is isometrically isomorphic to $(C_n, \|\cdot\|_{\mathrm{TC}})$, where $\|\cdot\|_{\mathrm{TC}}$ denotes the quotient norm of $(E_n, \|\cdot\|_1)/Z_n$.

Proposition 21. *Suppose $g_n^* \in (C_n, \|\cdot\|_\infty)$ satisfies*

$$g_n^*(g_n) = \|g_n\|_{\mathrm{TC}} \quad \textit{and} \quad g_n^*(g_j^i) = 0 \qquad (1 \leq j \leq n-1, 1 \leq i \leq 6^{n-j}).$$

Then $\|g_n^*\|_\infty \geq (4/3)^{n-1}$.

Proof. Note that $\|g_n\|_{\mathrm{TC}} = \|g_n\|_1$. This follows easily from convexity since each $h \in Z_n$ has a symmetric distribution relative to g_n (see Figure 6) and so $\|g_n + h\|_1 \geq \|g_n\|_1$. (In fact, one can show that $\|g_j^i\|_{\mathrm{TC}} = \|g_j^i\|_1$ for all i, j but this is not needed for the proof.) Note also that (see Figures 5 and 6)

$$\|f_n - \frac{1}{2} g_n\|_1 = \frac{3}{4}\|f_n\|_1.$$

Applying this to each sub-$\mathcal{L}_{n-1}$ of $\mathcal{L}_n$ (see Figure 6) gives

$$\|g_n - \frac{1}{2}\sum_i \varepsilon_{n-1}^i g_{n-1}^i\|_1 = \frac{3}{4}\|g_n\|_1$$

for some choice of signs $\varepsilon_{n-1}^i = \pm 1$. Repeating this argument, we get

$$\|g_n - \frac{1}{2}[\sum_i \varepsilon_{n-1}^i g_{n-1}^i + \frac{3}{2}\sum_i \varepsilon_{n-2}^i g_{n-2}^i]\|_1 = (\frac{3}{4})^2\|g_n\|_1$$

for some choice of $\varepsilon_j^i \in \{-1, 0, 1\}$. In general, we get for each $1 \leq k \leq n-1$,

$$\|g_n - \frac{1}{2}[\sum_{j=k}^{n-1}(\frac{3}{2})^{n-1-j}(\sum_i \varepsilon_j^i g_j^i)]\|_1 = (\frac{3}{4})^{n-k}\|g_n\|_1$$

for some choice of $\varepsilon_j^i \in \{-1, 0, 1\}$. Hence

$$\|g_n - \frac{1}{2}[\sum_{j=1}^{n-1}(\frac{3}{2})^{n-1-j}(\sum_i \varepsilon_j^i g_j^i)]\|_{\mathrm{TC}} \leq (\frac{3}{4})^{n-1}\|g_n\|_1 = (\frac{3}{4})^{n-1}\|g_n\|_{\mathrm{TC}}. \tag{11}$$

The desired result follows. □

Remark 22. The proof shows that the collection of special cut vectors g_j does not admit a bounded biorthogonal system (uniformly in n) for its span in $(C_n, \|\cdot\|_1)$. In particular, the orthogonal basis of C_n constructed above is not a Schauder basis (uniformly in n) in $(C_n, \|\cdot\|_1)$.

Moreover, (11) show that the equivalence constant of the basis of $\|\cdot\|_1$-normalized (or $\|\cdot\|_{\mathrm{TC}}$-normalized) special cut vectors with the unit vector basis of ℓ_1 is at least $(4/3)^{n-1}$.

On the other hand, the orthogonal basis of Z_n constructed above is a monotone Schauder basis for $(Z_n, \|\cdot\|_1)$. This allows an estimate from above for $d_{BM}(Z_n, \ell_1^N)$.

Proposition 23. *$d((Z_n, \|\cdot\|_1), \ell_1^N) \leq 2n$, where $N = \dim(Z_n) = (6^n - 1)/5$.*

Proof. For $1 \leq j \leq n$, let $H_j = (h_j^i)_{i=1}^{6^{n-j}}$ be an enumeration of the Z_n basis vectors of the form h_j for some sub-$\mathcal{L}_j$. Since each h_j^i is symmetric on its support sub-$\mathcal{L}_j$ it follows by convexity that $\cup_{j=0}^{n-1} H_{n-j}$ is a monotone basis of $(Z_n\|\cdot\|)$. Moreover, $\{h_j^i/\|h_j^i\|_1 \colon 1 \leq i \leq 6^{n-j}\}$ is 1-equivalent to the unit vector basis of $\ell_1^{6^j}$ since these vectors have disjoint supports. Let $x \in Z_n$

and write $x = \sum_{k=0}^{n-1} x_k$, where $x_k \in \operatorname{span}(H_{n-k})$. Then, by monotonicity of the basis,

$$\sum_{k=0}^{n-1} \|x_k\| \geq \|x\| \geq \frac{1}{2} \max_{0 \leq k \leq n-1} \|x_k\| \geq \frac{1}{2n} \sum_{k=0}^{n-1} \|x_k\|$$

Hence $\cup_{j=0}^{n-1} H_{n-j}$ is $2n$-equivalent to a suitably scaled standard basis of ℓ_1^n, which gives the result. □

6. Multi-branching diamond graphs

In this section we sharpen some of the results of [8, Section 6].

Theorem 24. *For each $k \geq 2$ and $n \geq 1$,*

$$\lambda(\mathrm{Lip}_0(D_{n,k})) = \frac{2k-2}{2k-1} n + \frac{4k^2 - 6k + 3}{(2k-1)^2} + \frac{2k-2}{(2k-1)^2} \frac{1}{(2k)^n}.$$

In particular, for $k = 2$ and $n \geq 1$,

$$\lambda(\mathrm{Lip}_0(D_n)) = \frac{2n}{3} + \frac{7}{9} + \frac{2}{9} 4^{-n}.$$

Proof. Let us recall the representation of $D_{n,k}$ used in [8]. We identify the edge space of $D_{n,k}$ with a subspace of $L_1[0,1]$ as follows. For $n = 1$ and $1 \leq j \leq k$ we identify the pair of edge vectors of the j^{th} path of length 2 from the 'top' to the 'bottom' vertex with the L_1-normalized indicator functions $2k1_{(j-1)/k,(2j-1)/(2k)]}$ and $2k1_{((2j-1)/(2k),j/k]}$. For $n \geq 2$, the edge space of $D_{n,k}$ is obtained from that of $D_{n,k-1}$ by subdividing the intervals corresponding to edge vectors of $D_{n,k-1}$ into $2k$ subintervals each of length $(2k)^{-n}$. Each of the k consecutive disjoint pairs of L_1-normalized indicator functions of the subintervals corresponds to each pair of edge vectors of the k paths of length 2 from the top and bottom vertices of the copy of $D_{1,k}$ which replaces the edge vector of $D_{n-1,k}$ corresponding to the interval of length $(2k)^{n-1}$ which is subdivided. We have now identified the edge vectors of $D_{n,k}$ with the L_1-normalized indicator functions

$$e_{n,j} = (2k)^n 1_{((j-1)/(2k)^n, j/(2k)^n]} \quad (1 \leq j \leq (2k)^n).$$

A basis for the cycle space corresponds to the L_∞-normalized system $\cup_{i=1}^n \{g_{i,j} : 1 \leq j \leq (2k)^{i-1}(k-1)\}$, where, setting $j = a(k-1) + b$ with $0 \leq a \leq (2k)^{i-1} - 1$ and $1 \leq b \leq k-1$,

$$g_{i,j} = (2k)^{-i}(e_{i,a2^k+2b-1} + e_{i,a2^k+2b} - e_{i,a2^k+2b+1} - e_{i,a2^k+2b+2}).$$

For $k \geq 3$, note that $g_{i,j}$ *overlaps* with $g_{i,j+1}$ when $b \leq k-2$, and hence this is not an orthogonal basis.

An orthogonal basis for the cut space corresponds to the L_∞-normalized system $\{h_0\} \cup \cup_{i=1}^n \{h_{i,j} : 1 \leq j \leq (2k)^i\}$, where $h_0 = 1_{[0,1]}$, and

$$h_{i,j} = (2k)^{-i}(e_{i,2j-1} - e_{i,2j}).$$

Let G be the group of automorphisms of the edge space generated by those automorphisms which interchange (by translations) the intervals $\{g_{i,j} > 0\}$ and $\{g_{i,j} < 0\}$ or the sets $\{h_{i,j} > 0\}$ and $\{h_{i,j} < 0\}$. Then (as observed in [8]) arguing as in Lemma 12, the orthogonal projection $P_{n,k}$ onto the cut space is the *unique* G-invariant projection onto the cut space. First, let us compute the $\|\cdot\|_1$- norm of $P_{n,k}$. Note that

$$P_{n,k}(e_{n,1}) = h_0 + \frac{1}{2}\sum_{i=1}^{n}(2k)^i h_{i,1}.$$

An elementary calculation which we omit yields

$$\|P_{n,k}(e_{n,1})\|_1 = \frac{2k-2}{2k-1}n + \frac{4k^2-6k+3}{(2k-1)^2} + \frac{2k-2}{(2k-1)^2}\frac{1}{(2k)^n}.$$

Now suppose $1 \le j \le (2n)^k$. For $1 \le i \le n$, let $\mathrm{supp}(e_{n,j}) \subset \mathrm{supp}(h_{i,r(i)})$. Then

$$P_{n,k}(e_{n,j}) = h_0 + \frac{1}{2}\sum_{i=1}^{n}\mathrm{sgn}(\langle e_{n,j}, h_{i,r(i)}\rangle)(2k)^i h_{i,r(i)}.$$

So $P_{n,k}(e_{n,j})$ has the same *distribution* as $P_{n,k}(e_{n,1})$. In particular, $\|P_{n,k}(e_{n,j})\|_1 = \|P_{n,k}(e_{n,1})\|_1$. Hence

$$\|P_{n,k}\|_1 = \max_{1\le j\le (2n)^k}\|P_{n,k}(e_{n,j})\|_1 = \|P_{n,k}(e_{n,1})\|_1.$$

Finally, since $P_{n,k}$ is the unique G-invariant projection onto the cut space and is self-adjoint,

$$\lambda(\mathrm{Lip}(D_{n,k})) = \|P_{n,k}\|_\infty = \|P_{n,k}\|_1 = \frac{2k-2}{2k-1}n + \frac{4k^2-6k+3}{(2k-1)^2} + \frac{2k-2}{(2k-1)^2}\frac{1}{(2k)^n}.$$

□

As a corollary, we get an improvement on [8, Theorem 6.10].

Corollary 25. *For each $n \ge 1$ and $k \ge 2$, the Banach-Mazur distance $d_{n,k}$ from the transportation cost space $\mathrm{TC}(D_{n,k})$ to the ℓ_1^N space of the same dimension satisfies*

$$d_{n,k} \ge \frac{2k-2}{2k-1}n + \frac{4k^2-6k+3}{(2k-1)^2} + \frac{2k-2}{(2k-1)^2}\frac{1}{(2k)^n}.$$

7. Characterization of finite trees in terms of their transportation cost spaces

The following result is well known.

Proposition 26. *Let M be a finite metric space with n elements. The space $\mathrm{TC}(M)$ is isometric to ℓ_1^{n-1} if and only if M is a weighted tree (the weight of an edge is the distance between its ends) with its shortest path distance.*

Apparently for finite metric spaces it is folklore. The earliest proof of the "if" part we are aware of is [10, Corollary 3.6]. Its more general version for infinite metric spaces was proved in [6]. Our goal is to give a direct proof of the "only if" part. A simple direct proof of the "if" part can be found in [8, Proposition 2.1].

Proof. We suppose that $\mathrm{TC}(M)$ is isometric to ℓ_1^{n-1} and prove that this implies that T is isometric to a weighted tree.

We may and shall identify the metric space M with a complete weighted graph, whose vertex set is M and for which the weight of an edge is the distance between its ends. In such a case the metric of M coincides with the weighted graph distance of this graph.

An edge uv in this weighted graph is called *essential* if and only if $d(u, v) < d(u, w) + d(w, v)$ for every $w \in M\backslash\{u, v\}$, or, equivalently, if the weighted graph distance of this graph will change if the edge uv is deleted.

It is well known (and easy to check) that for a finite metric space a vector f is an extreme point of the unit ball of $\mathrm{TC}(M)$ if and only if $f = (\mathbf{1}_u - \mathbf{1}_v)/d(u, v)$ for some essential edge uv in the described weighted graph (this result is known in a more general form [1], in which it is far from being easy).

Since ℓ_1^{n-1} has $(n-1)$ symmetric pairs of extreme points, we conclude that the weighted graph corresponding to M has $(n-1)$ essential edges. Since it is clear that the set of essential edges has to connect the graph, we get that the set of essential edges in M forms a spanning tree. Recalling the definition of essential edges, we derive that the metric of M is the distance of the weighted tree formed by essential edges. □

Corollary 27. *The space* $\mathrm{TC}(M)$ *with* $|M| = n$ *has between* $(n-1)$ *and* $\frac{n(n-1)}{2}$ *symmetric pairs of extreme points and thus is a quotient of* ℓ_1^d *for* $(n-1) \leq d \leq \frac{n(n-1)}{2}$.

Proof. In fact, the number of essential edges in a weighted connected simple graph with n vertices can be any number between $(n-1)$ and $\frac{n(n-1)}{2}$. This follows from the following easy observations: (a) All edges in an unweighted (equivalently, a weighted graph with all weights equal to 1) connected simple graph are essential, and the number of such edges can be any number between $(n-1)$ and $\frac{n(n-1)}{2}$. (b) Essential edges induce a connected spanning graph, and thus there should be at least $(n-1)$ of them. □

8. Isometric copies of ℓ_∞^3 and ℓ_∞^4 in $\mathrm{TC}(M)$ on finite metric spaces

One of the results of [17] is a construction of finite metric spaces for which $\mathrm{TC}(M)$ contains isometric copies of ℓ_∞^3 and ℓ_∞^4. The goal of this last section is to provide a simpler constructions of such spaces. We show that

(1) There exists a 6-point set T such that $\mathrm{TC}(T)$ contains ℓ_∞^3 isometrically.
(2) There exists an 8-point set F such that $\mathrm{TC}(F)$ contains ℓ_∞^4 isometrically.

Below we describe the metric spaces and the transportation problems spanning ℓ_∞^3 and ℓ_∞^4, respectively. We leave it as an exercise the straightforward verification of the equality

$$\left\|\sum_{i=1}^{k}\theta_i f_i\right\| = 1$$

for $k = 3$ or $k = 4$, and $\theta_i = \pm 1$.

The description of the metric space T:

	a	b	c	d	e	f
a	0	1	1	1	1/2	1/2
b	1	0	1	1	1/2	1/2
c	1	1	0	1	1/2	1/2
d	1	1	1	0	1/2	1/2
e	1/2	1/2	1/2	1/2	0	1
f	1/2	1/2	1/2	1/2	1	0

Table 1. Distances

The description of three transportation problems on T spanning ℓ_∞^3:

	a	b	c	d	e	f
f_1	1/2	-1/2	1/2	-1/2	0	0
f_2	1/2	1/2	-1/2	-1/2	0	0
f_3	0	0	0	0	1	-1

Table 2. Values of transportation problems

The description of the metric space F:

	a	b	c	d	e	f	g	h
a	0	1	1	1	1/2	1/2	1/2	1/2
b	1	0	1	1	1/2	1/2	1/2	1/2
c	1	1	0	1	1/2	1/2	1/2	1/2
d	1	1	1	0	1/2	1/2	1/2	1/2
e	1/2	1/2	1/2	1/2	0	1	1	1
f	1/2	1/2	1/2	1/2	1	0	1	1
g	1/2	1/2	1/2	1/2	1	1	0	1
h	1/2	1/2	1/2	1/2	1	1	1	0

Table 3. Distances

The description of four transportation problems on F spanning ℓ_∞^4:

	a	b	c	d	e	f	g	h
f_1	1/2	-1/2	1/2	-1/2	0	0	0	0
f_2	1/2	1/2	-1/2	-1/2	0	0	0	0
f_3	0	0	0	0	1/2	-1/2	1/2	-1/2
f_4	0	0	0	0	1/2	1/2	-1/2	-1/2

Table 4. Values of transportation problems

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References

[1] R. J. Aliaga, E. Pernecká, Supports and extreme points in Lipschitz-free spaces, *Rev. Mat. Iberoam.* (2020), in press, doi:10.4171/rmi/1191.

[2] A. D. Andrew, *On subsequences of the Haar system in* $C(\Delta)$, Israel J. Math. 31 (1978), 85–90.

[3] F. P. Baudier, P. Motakis, T. Schlumprecht, A. Zsák, Stochastic approximation of lamplighter metrics, arXiv:2003.06093v2.

[4] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis.* Vol. **1**. American Mathematical Society Colloquium Publications, **48**. American Mathematical Society, Providence, RI, 2000.

[5] M. S. Charikar, Similarity estimation techniques from rounding algorithms. *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, 380–388, ACM, New York, 2002.

[6] A. Dalet, P.L. Kaufmann, A. Procházka, Characterization of metric spaces whose free space is isometric to ℓ_1. *Bull. Belg. Math. Soc. Simon Stevin* **23** (2016), no. 3, 391–400.

[7] R. Diestel, *Graph theory*, Fifth edition, Graduate Texts in Mathematics, **173**, Springer-Verlag, Berlin, 2017.

[8] S. J. Dilworth, Denka Kutzarova, and Mikhail I. Ostrovskii, *Lipschitz-free Spaces of Finite Metric Spaces*, Canad. J. Math. 72 (2020), 774–804.

[9] J. Fakcharoenphol, S. Rao, K. Talwar, A tight bound on approximating arbitrary metrics by tree metrics, *J. Comput. System Sci.*, **69** (2004), no. 3, 485–497; Conference version: *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, 448–455, ACM, New York, 2003.

[10] A. Godard, Tree metrics and their Lipschitz-free spaces, *Proc. Amer. Math. Soc.*, **138** (2010), no. 12, 4311–4320.

[11] B. Grünbaum, *Projection constants*, Trans. Amer. Math. Soc. 95 (1960) 451–465.

[12] A. Gupta, I. Newman, Y. Rabinovich, A. Sinclair, Cuts, trees and ℓ_1-embeddings of graphs, *Combinatorica*, **24** (2004) 233–269; Conference version in: *40th Annual IEEE Symposium on Foundations of Computer Science*, 1999, pp. 399–408.

[13] P. Indyk, N. Thaper, Fast image retrieval via embeddings, in: *ICCV 03: Proceedings of the 3rd International Workshop on Statistical and Computational Theories of Vision*, 2003.

[14] W.B. Johnson and G. Schechtman, *Diamond graphs and super-reflexivity*, J. Topol. Anal., 1 (2009), no. 2, 177-189.

[15] L. V. Kantorovich, On mass transportation (Russian), *Doklady Acad. Naus SSSR*, (N.S.) **37**, (1942), 199–201; English transl.: *J. Math. Sci.* (N. Y.), **133** (2006), no. 4, 1381–1382.

[16] L. V. Kantorovich, M. K. Gavurin, Application of mathematical methods in the analysis of cargo flows (Russian), in: *Problems of improving of transport efficiency*, USSR Academy of Sciences Publishers, Moscow, 1949, pp. 110–138.

[17] S. S. Khan, M. Mim, M. I. Ostrovskii, Isometric copies of ℓ_∞^n and ℓ_1^n in transportation cost spaces on finite metric spaces, in: *The Mathematical Legacy of Victor Lomonosov. Operator Theory*, pp. 189–203, De Gruyter, 2020; DOI:10.1515/9783110656756-014.

[18] T. J. Laakso, Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincare inequality, *Geom. Funct. Anal.*, **10** (2000), no. 1, 111–123.

[19] Urs Lang and Conrad Plaut, *Bilipschitz Embeddings of Metric Spaces into Space Forms*, Geom. Dedicata 87 (2001), 285–307.

[20] J.R. Lee and P. Raghavendra, *Coarse differentiation and multi-flows in planar graphs*, Discrete Comput. Geom. 43 (2) (2010), 346–362.

[21] Sofiya Ostrovska and Mikhail I. Ostrovskii, *Non-existence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs*, Discrete Math. 340 (2017), 9–17.

[22] S. Ostrovska, M. I. Ostrovskii, Generalized transportation cost spaces, *Mediterr. J. Math.* **16** (2019), no. 6, Paper No. 157.

[23] S. Ostrovska, M. I. Ostrovskii, On relations between transportation cost spaces and ℓ_1, *J. Math. Anal. Appl.*, **491** (2020), no. 2, 124338, https://doi.org/10.1016/j.jmaa.2020.124338.

[24] M. I. Ostrovskii, On metric characterizations of some classes of Banach spaces, *C. R. Acad. Bulgare Sci.*, **64** (2011), no. 6, 775–784.

[25] M. I. Ostrovskii, *Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces*, de Gruyter Studies in Mathematics, **49**. Walter de Gruyter & Co., Berlin, 2013.

[26] M. I. Ostrovskii and B. Randrianantoanina, *A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces*, J. Funct. Anal. 273 (2017), no. 2, 598–651.

[27] W. Rudin, *Projections on invariant subspaces*, Proc. Amer. Math. Soc. 13 (1962), 429–432.

[28] N. Weaver, *Lipschitz algebras*, Second edition, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

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