MTH 211: HW#5 SOLUTIONS

Chapter 6

10. Let \( G \) be a group. Prove that the mapping \( \alpha(g) = g^{-1} \) for all \( g \) in \( G \) is an automorphism if and only if \( G \) is abelian.

Proof. It is easy to see that \( \alpha \) is a bijection. Therefore, we will prove that \( \alpha \) is a homomorphism if and only if \( G \) is abelian.

\[
\alpha(gh) = \alpha(g)\alpha(h), \; \forall g, h \in G \iff (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1}, \; \forall g, h \in G
\]

\[
\iff gh = hg, \; \forall g, h \in G.
\]

\[\square\]

22. Prove or disprove that \( U(20) \) and \( U(24) \) are isomorphic.

Proof. \( U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\} \) and \( U(24) = \{1, 5, 7, 11, 13, 17, 19, 23\} \). Every non-trivial element of \( U(24) \) has order 2, but this is not true for \( U(20) \). For example, \( |3| = 4 \) in \( U(20) \). Therefore, \( U(20) \) and \( U(24) \) are not isomorphic. \[\square\]

36. Let \( G = \{0, \pm2, \pm4, \pm6, \ldots\} \) and \( H = \{0, \pm3, \pm6, \pm9, \ldots\} \). Show that \( G \) and \( H \) are isomorphic groups under addition. Does your isomorphism preserve multiplication?

Proof. Notice that \( G = \langle 2 \rangle \) and \( H = \langle 3 \rangle \). Let \( \phi : G \to H \) be the function defined by, \( \phi(x) = \frac{3}{2}x \). It is straightforward to check that \( \phi \) is a bijection and a homomorphism under addition. That is, \( \phi \) is an isomorphism under addition. However, this function does not preserve multiplication. For example, \( \phi(2) \cdot \phi(4) = 3 \cdot 6 = 18 \neq 12 = \phi(8) = \phi(2 \cdot 4) \). \[\square\]