Abstract: This is a collection of problems suggested at two meetings of the seminar “Nonlinear geometry of Banach spaces” which took place during the 2009 Workshop in Analysis and Probability at Texas A & M University. Participants of the seminar: F. Baudier, J. Chavez-Dominguez, D. Dosev, T. Figiel, W.B. Johnson, P.W. Nowak, M.I. Ostrovskii, B. Randrianantoanina, B. Sari, G. Schechtman.

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1 First meeting (August 5, 2009)

Problem 1.1 (W. B. Johnson) Find a purely metric characterization of reflexivity or Radon-Nikodým property.

Comment (02/13/2011). It is known ([CK09, Corollary 1.7], [Ost11, Theorem 3.6] that the Laakso [Laa00, Laa02] space does not admit a bilipschitz embeddings into a Banach space with the Radon-Nikodým property, but the converse does not seem to hold (see [Ost11]). Also, it is known that the infinite diamond does not admit a bilipschitz embedding into a Banach space with the Radon-Nikodým property, but the converse does not hold [Ost11, Corollary 3.3].

Comment (04/04/2014). Reflexivity was characterized in [Ost14+a] in terms of bilipschitz embeddings of ‘submetric’ spaces introduced in that paper.

An ‘if and only if’ metric characterization of the Radon-Nikodým property was found in [Ost14]. The characterization is terms of thick families of geodesics defined as follows. Let $u$ and $v$ be two elements in a metric space $(M, d_M)$. A family $T$ of $uv$-geodesics is called thick if there is $\alpha > 0$ such that for every $g \in T$ and for every finite collection of points $r_1, \ldots, r_n$ in the image of $g$, there is another $uv$-geodesic $\tilde{g}$ satisfying the conditions:
(1) The image of $\tilde{g}$ also contains $r_1, \ldots, r_n$. (2) Possibly there are some more common points of $g$ and $\tilde{g}$. (3) We can find a sequence $0 < s_1 < q_1 < s_2 < q_2 < \cdots < s_m < q_m < s_{m+1} < d_M(u, v)$, such that $g(q_i) = \tilde{g}(q_i)$ ($i = 1, \ldots, m$) are common points containing $r_1, \ldots, r_n$, and the images $g(s_i)$ and $\tilde{g}(s_i)$ are distinct and the sum of deviations over them is nontrivially large in the sense that $\sum_{i=1}^{m+1} d_M(g(s_i), \tilde{g}(s_i)) \geq \alpha$.

The characterization of [Ost14] is: A Banach space $X$ does not have the Radon-Nikodym property if and only if there exists a metric space $M_X$ containing a thick family $T_X$ of geodesics which admits a bilipschitz embedding into $X$.

On the other hand, it was shown in [Ost14+a] that for each thick family of geodesics there is a non-RNP Banach space which does not admit a uniformly bilipschitz embedding of it.

In addition, in [Ost14+b] it was shown that a geodesic metric space which does not admit bilipschitz embeddings into Banach spaces with the Radon-Nikodym property does not necessarily contain a bilipschitz image of a thick family of geodesics. More precisely, the Heisenberg group with its subriemannian metric is an example of such metric space.

2 Second meeting (August 10, 2009)

2.1 Ostrovskii's talk and problems

The main motivation for the results of the talk is the problem: Let $M$ be a metric space with bounded geometry which is not coarsely embeddable into a Hilbert space. Does it follow that $M$ contains weakly a sequence of expanders (see [GK04], [Ost09a], [Tes09])?

Comment (04/04/2014). This problem was solved in the negative by Arzhantseva and Tessera [AT14] using relative property (T) and the corresponding property of relative expansion.

Talk was based on [Ost09b] where expansion properties of structures discovered in [Ost09a] and [Tes09] were studied.

At the end of the talk the following problems were suggested:

(1) Suppose that we have a sequence $\{G_n\}$ of graphs of uniformly bounded degrees and $h > 0$ (which does not depend on $n$) such that in each $G_n$, sets of diameters $\leq n$ are $h$-expanding. (The number of vertices in $G_n$ and its diameter can be significantly larger than $n$.) Can we find in it a substructures $H_n \subset G_n$ which are weak expanders?

Remark. One of the classes of graphs satisfying the described condition are families of graphs of fixed degree $\geq 3$ and indefinitely growing girth.

(2) Let $\{G_n\}_{n=1}^\infty$ be a family of graphs satisfying the conditions of (1). Does it follow that $\{G_n\}$ are not uniformly coarsely embeddable into $L_1$?

Comment (09/05/2011). Both problems were (implicitly) solved in the negative in [AGS12]. See also [Ost12] for more on the construction of [AGS12].
2.2 Johnson’s modifications of problems from Section 2.1

(a) What kind of expansion of metric spaces/graphs implies that they cannot be uniformly coarsely embedded into a Hilbert space?
(b) Is there a sequence of finite metric spaces whose uniformly coarse embeddability into a metric space $M$ is equivalent to the fact that $M$ is not coarsely embeddable into a Hilbert space?

2.3 Johnson’s problems on characterization of spaces with no cotype

1. Find a sequence of graphs with uniformly bounded degrees whose uniformly bilipschitz embeddability into a Banach space $X$ is equivalent to the statement: $X$ has no cotype. The same for type.

Comment (02/13/2011). Problem 1 was solved in [Ost11].

Comment (09/05/2011). See [Ost13] for further results in the same direction.

2. Find a sequence of spaces with bounded geometry whose uniformly coarse embeddability into a Banach space $X$ is equivalent to the statement: $X$ has no cotype.

2.4 Arens-Eells spaces on expanders

(M.I. Ostrovskii) Let $\{G_n\}$ be a family of expanders and $\{A_n\}$ be the sequence of the corresponding Arens-Eells spaces. Is it true that $\{A_n\}$ contain $\ell^k_\infty$ of growing dimensions?

See [Kal08] and [Wea99] for basic facts on Arens-Eells spaces.

References


