Metric properties of expanders
Part 5: Expansion properties of metric spaces not admitting a coarse embedding into a Hilbert space

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In these notes OME means M. I. Ostrovskii, *Metric Embeddings*, book in preparation
Let \((M, d_M)\) be a locally finite metric space which is not coarsely embeddable into \(L_1\). Such spaces were characterized in [Ost09] and [Tes09]. We reproduce the characterization in the form in which it appears in [Ost09].
Let \((M, d_M)\) be a locally finite metric space which is not coarsely embeddable into \(L_1\). Such spaces were characterized in [Ost09] and [Tes09]. We reproduce the characterization in the form in which it appears in [Ost09].

**Theorem ([Ost09, Theorem 2.4])**

Let \(M\) be a locally finite metric space which is not coarsely embeddable into \(L_1\). Then there exists a constant \(D\), depending on \(M\) only, such that for each \(n \in \mathbb{N}\) there exists a finite set \(M_n \subset M\) and a probability measure \(\mu_n\) on \(M_n \times M_n\) such that

\[
d_M(u, v) \geq n \text{ for each } (u, v) \in \text{supp} \mu_n.
\]

For each Lipschitz function \(f : M \to L_1\) we have

\[
\int_{M_n \times M_n} ||f(u) - f(v)||_{L_1} \, d\mu_n(u, v) \leq D\text{Lip}(f). \tag{1}
\]
Our first purpose is to discover some expansion properties of sets $M_n$. 
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Let $s$ be a positive integer. We consider graphs $G(n, s) = (M_n, E(M_n, s))$, where the edge set $E(M_n, s)$ is obtained by joining those pairs of vertices of $M_n$ which are at distance $\leq s$. The graphs $G(n, s)$ have uniformly bounded degrees if the metric space $M$ has bounded geometry.
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**Observation:** Each vertex cut of $G(n, s)$ separates it into pieces with $d_M$-distance between them at least $s$. 
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(*) For some $s \in \mathbb{N}$ there is a number $h_s > 0$ and subgraphs $H_n$ of $G(n, s)$ of indefinitely growing sizes (as $n \to \infty$) such that the expansion constants of $\{H_n\}$ are uniformly bounded from below by $h_s$. 
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If we would prove that in the bounded geometry case the condition (*) is satisfied, it would solve the well-known problem (see [GK04], [Ost09], [Tes09]): whether each metric space with bounded geometry which does not embed coarsely into a Hilbert space contains weak expanders? For spaces with bounded geometry weak expanders are defined as Lipschitz images $f_m(X_m)$ of (vertex sets) of a family of expanders with uniformly bounded Lipschitz constants of $\{f_m\}_{m=1}^{\infty}$ and without dominating pre-images in the sense that 

$$\lim_{m \to \infty} \max_{z \in f_m(X_m)} \left( \frac{|f_m^{-1}(z)|}{|X_m|} \right) = 0.$$
At this point we are able to prove only the following weaker expansion property of the graphs $G(n, s)$. We introduce the measure $\nu_n$ on $M_n$ by $\nu_n(A) = \mu_n(A \times M_n)$. Let $F$ be an induced subgraph of $G(n, s)$. We denote the vertex boundary of a set $A$ of vertices in $F$ by $\delta_F A$. 

Theorem

Let $s$ and $n$ be such that $2n > s > 8D$. Let $\phi(D, s) = \frac{s}{4D} - 2$. Then $G(n, s)$ contains an induced subgraph $F$ with $d_{M}$-diameter $\geq n - \frac{s}{2}$, such that each subset $A \subset F$ of $d_{M}$-diameter $< n - \frac{s}{2}$ satisfies the condition:

$$\nu_n(\delta_F A) > \phi(D, s) \nu_n(A).$$
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Proof of Theorem 2. Suppose that for some \( n, s \in \mathbb{N} \) satisfying \( 2n > s > 8D \) there is no such subgraph in \( G(n, s) \). Then for each induced subgraph \( F \) in \( G(n, s) \) of \( d_M \)-diameter \( \geq n - \frac{s}{2} \) we can find a subset \( A \subset F \) of \( d_M \)-diameter \( < n - \frac{s}{2} \) such that \( \nu_n(\delta_F A) \leq \varphi(D, s)\nu_n(A) \). We start with \( F_1 = G(n, s) \) (the definitions of \( M_n \) and \( \mu_n \) imply that the \( d_M \)-diameter of \( M_n \) is \( \geq n \)), find such \( A_1 \subset F_1 \) and remove \( A_1 \cup \delta_{F_1} A_1 \) from \( G(n, s) \). If the obtained graph \( F_2 \) still has \( d_M \)-diameter \( \geq n - \frac{s}{2} \), we find a subset \( A_2 \) in it such that \( \nu_n(\delta_{F_2} A_2) \leq \varphi(D, s)\nu_n(A_2) \). We remove the subset \( A_2 \cup \delta_{F_2} A_2 \) from \( F_2 \). We continue in an obvious way till we get a set of \( d_M \)-diameter \( < n - \frac{s}{2} \) (this should eventually happen since \( M_n \) is finite). We denote this set \( A_p \), where \( p \) is the number of steps in the process.
Proof of Theorem 2. Suppose that for some \( n, s \in \mathbb{N} \) satisfying \( 2n > s > 8D \) there is no such subgraph in \( G(n, s) \). Then for each induced subgraph \( F \) in \( G(n, s) \) of \( d_M \)-diameter \( \geq n - \frac{s}{2} \) we can find a subset \( A \subset F \) of \( d_M \)-diameter \( < n - \frac{s}{2} \) such that \( \nu_n(\delta_F A) \leq \varphi(D, s)\nu_n(A) \). We start with \( F_1 = G(n, s) \) (the definitions of \( M_n \) and \( \mu_n \) imply that the \( d_M \)-diameter of \( M_n \) is \( \geq n \)), find such \( A_1 \subset F_1 \) and remove \( A_1 \cup \delta_{F_1} A_1 \) from \( G(n, s) \). If the obtained graph \( F_2 \) still has \( d_M \)-diameter \( \geq n - \frac{s}{2} \), we find a subset \( A_2 \) in it such that \( \nu_n(\delta_{F_2} A_2) \leq \varphi(D, s)\nu_n(A_2) \). We remove the subset \( A_2 \cup \delta_{F_2} A_2 \) from \( F_2 \). We continue in an obvious way till we get a set of \( d_M \)-diameter \( < n - \frac{s}{2} \) (this should eventually happen since \( M_n \) is finite). We denote this set \( A_p \), where \( p \) is the number of steps in the process.

Observe that each of the sets \( A_i \) has diameter \( < n - \frac{s}{2} \), and that the \( d_M \)-distance between any \( A_i \) and \( A_j \) \((i \neq j)\) is at least \( s \) (see the observation above).
We introduce a family of 1-Lipschitz functions $f_\theta$ on $M$, where $\theta = \{\theta_i\}_{i=1}^p \in \Theta = \{-1, 1\}^p$ by the formula:

$$f_\theta(x) = \begin{cases} 
\theta_j \left( \frac{s}{2} - \text{dist}(x, A_j) \right) & \text{if } \text{dist}(x, A_j) < \frac{s}{2} \\
0 & \text{if } \text{dist}(x, \bigcup_{i=1}^p A_i) \geq \frac{s}{2}.
\end{cases}$$

The function is well-defined since the inequality $\text{dist}(x, A_j) < \frac{s}{2}$ cannot be satisfied for more than one value of $j$. Straightforward verification shows that this function is 1-Lipschitz.
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We endow $\Theta = \{-1, 1\}^p$ with the natural probability measure $\mathcal{P}$ and introduce for each $x \in M$ a function $F_x \in L_1(\Theta, \mathcal{P})$ given by $F_x(\theta) = f_\theta(x)$. It is clear that the mapping $x \mapsto F_x$ is 1-Lipschitz.
Applying inequality (1) to this mapping we get

\[ D \geq \int_{M_n \times M_n} \left\| F_x(\theta) - F_y(\theta) \right\|_{L_1(\Theta, \mathcal{P})} d\mu_n(x, y) \]

\[ \geq \int_{M_n \times M_n} \int_{\Theta} |f_\theta(x) - f_\theta(y)|d\mathcal{P}(\theta)d\mu_n(x, y) \]

\[ \geq \int_{M_n \times M_n} \int_{\Psi(x, y)} |f_\theta(x)|d\mathcal{P}(\theta)d\mu_n(x, y), \]

where \( \Psi(x, y) \) is the subset of \( \Theta \) for which \( f_\theta(x) \) and \( f_\theta(y) \) have different signs.
Applying inequality (1) to this mapping we get

\[ D \geq \int_{M_n \times M_n} \| F_x(\theta) - F_y(\theta) \|_{L_1(\Theta, \mathcal{P})} d\mu_n(x, y) \]

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\[ \geq \int_{M_n \times M_n} \int_{\psi(x, y)} |f_\theta(x)| d\mathcal{P}(\theta) d\mu_n(x, y), \]

where \( \psi(x, y) \) is the subset of \( \Theta \) for which \( f_\theta(x) \) and \( f_\theta(y) \) have different signs.

Observe that the value of \( |f_\theta(x)| \) does not depend on \( \theta \). We get

\[ \int_{M_n \times M_n} \int_{\psi(x, y)} |f_\theta(x)| d\mathcal{P}(\theta) d\mu_n(x, y) \]

\[ \geq \int_{(\bigcup_{i=1}^p A_i) \times M_n} |f_\theta(x)| \int_{\psi(x, y)} d\mathcal{P}(\theta) d\mu_n(x, y). \]
Now we observe that for $x \in A_j$ and $y$ satisfying $(x, y) \in \text{supp} \mu_n$ we have $d_M(x, y) \geq n$ and therefore $d_M(y, A_j) \geq \frac{s}{2}$ (recall that the diameter of $A_j$ is $< n - \frac{s}{2}$). Hence $\mathcal{P}(\Psi(x, y)) \geq \frac{1}{2}$ for each pair $(x, y)$ from $\text{supp} \mu_n$. 

\[ \int \bigcup_{i=1}^{p} A_i \times M_n |f\theta(x)| \int \Psi(x, y) \, d\mathcal{P}(\theta) \, d\mu_n(x, y) \geq \int \bigcup_{i=1}^{p} A_i \times M_n s/4 \nu_n \bigcup_{i} A_i. \] 

If we recall the beginning of this chain of inequalities, we get $D \geq \frac{s}{4} \nu_n \bigcup_{i} A_i \cdot (2)$. 

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Now we observe that for \( x \in A_j \) and \( y \) satisfying \((x, y) \in \text{supp} \mu_n \) we have \( d_M(x, y) \geq n \) and therefore \( d_M(y, A_j) \geq \frac{s}{2} \) (recall that the diameter of \( A_j \) is \(< n - \frac{s}{2} \)). Hence \( P(\Psi(x, y)) \geq \frac{1}{2} \) for each pair \((x, y)\) from \( \text{supp} \mu_n \).

We get
\[
\int_{(\bigcup_{i=1}^p A_i) \times M_n} |f_\theta(x)| \int_{\Psi(x, y)} dP(\theta) d\mu_n(x, y) \geq \int_{(\bigcup_{i=1}^p A_i) \times M_n} \frac{s}{2} \cdot \frac{1}{2} d\mu_n(x, y) = \frac{s}{4} \nu_n(\bigcup_i A_i).
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Now we observe that for $x \in A_j$ and $y$ satisfying $(x, y) \in \text{supp} \mu_n$ we have $d_M(x, y) \geq n$ and therefore $d_M(y, A_j) \geq \frac{s}{2}$ (recall that the diameter of $A_j$ is $< n - \frac{s}{2}$). Hence $\mathcal{P}(\psi(x, y)) \geq \frac{1}{2}$ for each pair $(x, y)$ from $\text{supp} \mu_n$.

We get

$$
\int_{\left( \bigcup_{i=1}^{p} A_i \right) \times M_n} |f_\theta(x)| \int_{\psi(x,y)} d\mathcal{P}(\theta) d\mu_n(x, y)
\geq \int_{\left( \bigcup_{i=1}^{p} A_i \right) \times M_n} \frac{s}{2} \cdot \frac{1}{2} d\mu_n(x, y) = \frac{s}{4} \nu_n \left( \bigcup_i A_i \right).
$$

If we recall the beginning of this chain of inequalities, we get

$$
D \geq \frac{s}{4} \nu_n \left( \bigcup_i A_i \right). \tag{2}
$$
Observe that $\nu_n(\bigcup_i A_i) + \nu_n(\bigcup_i \delta_{F_i} A_i) = 1$ (since for each next set we consider the vertex boundary in the “remaining” graph, the boundaries are disjoint) and
$\nu_n(\bigcup_i \delta_{F_i} A_i) \leq \varphi(D, s) \nu_n(\bigcup_i A_i)$. Therefore

\[(1 + \varphi(D, s))\nu_n(\bigcup_i A_i) \geq 1 \quad (3)\]

Combining (2) and (3) we get

\[D \geq \frac{s}{4(1 + \varphi(D, s))}\]

and $\varphi(D, s) \geq \frac{s}{4D} - 1$, a contradiction. \qed
