Nonexistence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs

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Abstract

Diamond graphs and Laakso graphs are important examples in the theory of metric embeddings. Many results for these families of graphs are similar to each other. In this connection, it is natural to ask whether one of these families admits uniformly bilipschitz embeddings into the other. The well-known fact that Laakso graphs are uniformly doubling but diamond graphs are not, immediately implies that diamond graphs do not admit uniformly bilipschitz embeddings into Laakso graphs. The main goal of this paper is to prove that Laakso graphs do not admit uniformly bilipschitz embeddings into diamond graphs.

Keywords. diamond graphs, doubling metric space, Laakso space, Lipschitz map

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1 Introduction

Diamond graphs and Laakso graphs are important examples in the theory of metric embeddings, see [2, 7, 8, 9, 10, 11, 12, 13, 14] . Many results for these families of graphs are similar to each other, see the example after Definitions 1.1 and 1.2 and in Section 3. In this connection, the question emerges: does one of these families admit
uniformly bilipschitz embeddings into the other? The well-known fact that Laakso graphs are uniformly doubling but diamond graphs are not uniformly doubling - see Definition 1.3 and the subsequent discussion - immediately implies that diamond graphs do not admit uniformly bilipschitz embeddings into Laakso graphs. The main goal of this paper is to prove that Laakso graphs do not admit uniformly bilipschitz embeddings into diamond graphs.

To the best of our knowledge, the first paper in which diamond graphs \( \{D_n\}_{n=0}^{\infty} \) were used in Metric Geometry is the conference version of [4], which was published in 1999.

**Definition 1.1.** Diamond graphs \( \{D_n\}_{n=0}^{\infty} \) are defined recursively: The diamond graph of level 0 has two vertices joined by an edge of length 1 and is denoted by \( D_0 \). The diamond graph \( D_n \) is obtained from \( D_{n-1} \) in the following way. Given an edge \( uv \in E(D_{n-1}) \), it is replaced by a quadrilateral \( u, a, v, b \), with edges \( ua, av, vb, bu \). (See Figure 1.)

Two different normalizations of the graphs \( \{D_n\}_{n=1}^{\infty} \) can be found in the literature:

- **Unweighted diamonds:** Each edge has length 1.
- **Weighted diamonds:** Each edge of \( D_n \) has length \( 2^{-n} \).

In both cases, we endow the vertex sets of \( \{D_n\}_{n=0}^{\infty} \) with their shortest path metrics.
For weighted diamonds, the identity map $D_{n-1} \mapsto D_n$ is an isometry and, in this case, the union of $D_n$ endowed with the metric induced from $\{D_n\}_{n=0}^\infty$ is called the infinite diamond and denoted by $D_\omega$.

Another family of graphs considered in the present article is that of Laakso graphs. The Laakso graphs were introduced in [8], but they were inspired by the construction of Laakso [7].

Definition 1.2. Laakso graphs $\{L_n\}_{n=0}^\infty$ are defined recursively: The Laakso graph of level 0 has two vertices joined by an edge of length 1 and is denoted $L_0$. The Laakso graph $L_n$ is obtained from $L_{n-1}$ according to the following procedure. Each edge $uv \in E(L_{n-1})$ is replaced by the graph $L_1$ exhibited in Figure 2, the vertices $u$ and $v$ are identified with the vertices of degree 1 of $L_1$.

Similarly to the case of diamond graphs, the two different normalizations of the graphs $\{L_n\}_{n=1}^\infty$ are used:

- **Unweighted Laakso graphs**: Each edge has length 1.
- **Weighted Laakso graphs**: Each edge of $L_n$ has length $4^{-n}$.

In both situations, we endow vertex sets of $\{L_n\}_{n=0}^\infty$ with their shortest path metrics. In the case of weighted Laakso graphs, the identity map $L_{n-1} \mapsto L_n$ is an isometry and the union of $L_n$, endowed with the metric induced from $\{L_n\}_{n=0}^\infty$, is called the Laakso space and denoted by $L_\omega$. 
Many known results for one of the aforementioned families admit analogues for the other. For example, it is known [6] that both of the families can be used to characterize superreflexivity. Further, the two families consist of planar graphs with poor embeddability into Hilbert space, see [7, 8, 11]. Moreover, in many situations, the proofs used for one of the families can be easily adjusted to work for the other family. For instance, this is the case for the Markov convexity (see [10, Section 3] and Section 3 below). There are similarities between results for the infinite diamond and the Laakso space, too. For example, neither of spaces \( D_\omega \) and \( L_\omega \) admits bilipschitz embeddings into any Banach space with the Radon-Nikodým property [3, 12].

On the other hand, the families \( \{D_n\} \) and \( \{L_n\} \) are not alike in some important metrical respects, and the corresponding properties of the Laakso graphs were among the reasons for their introduction. To exemplify the differences, it can be mentioned that the Laakso graphs are uniformly doubling (see [8, Theorem 2.3]) and unweighted Laakso graphs have uniformly bounded geometry, whereas diamond graphs do not possess any of these properties. These facts are well known. Nevertheless, for the convenience of the readers, they will be proved after recalling the necessary definitions as a suitable reference is not available.

**Definition 1.3.** (i) A metric space \( X \) is called **doubling** if there is a constant \( D \geq 1 \) such that every bounded set \( B \) in \( X \) can be covered by at most \( D \) sets of diameter \( \text{diam}(B)/2 \). A sequence of metric spaces is called **uniformly doubling** if all of them are doubling and the constant \( D \) can be chosen to be the same for all of them. (ii) A metric space \( X \) is said to have **bounded geometry** if there is a function \( M : (0, \infty) \to (0, \infty) \) such that each ball of radius \( r \) in \( X \) has at most \( M(r) \) elements. A sequence of metric spaces is said to have **uniformly bounded geometry** if the function \( M(r) \) can be chosen to be the same for all of them.

It is easy to verify that a sequence of unweighted graphs has uniformly bounded geometry if and only if they have uniformly bounded degrees. This observation implies immediately that unweighted \( \{L_n\} \) has uniformly bounded geometry, but \( \{D_n\} \) does not.

The fact that \( \{D_n\} \) is not uniformly doubling - it is easy to see that this statement does not depend on normalization - can be shown in the following way. Consider unweighted diamonds and call one of the vertices of \( D_0 \) the top and the other the bottom. Define the top and the bottom of \( D_n \) as vertices which evolved from the top and the bottom of \( D_0 \), respectively. Now, consider the ball of radius 1 centered at the bottom of \( D_n \). It contains \( 2^n + 1 \) elements (including the bottom) and has diameter 2. On the other hand, it can be readily seen that any subset of this ball of diameter 1 contains two elements. The fact that \( \{D_n\} \) is not uniformly doubling follows. Many interesting results on doubling metric spaces can be found in [5].

Our goal is to show that the diamond and Laakso graphs are not alike in one more respect in terms of the next standard definition:
Definition 1.4. (i) Let $0 \leq C < \infty$. A map $f : (A,d_A) \to (Y,d_Y)$ between two metric spaces is called $C$-Lipschitz if
\[
\forall u,v \in A \quad d_Y(f(u), f(v)) \leq Cd_A(u,v).
\]
A map $f$ is called Lipschitz if it is $C$-Lipschitz for some $0 \leq C < \infty$.

(ii) Let $1 \leq C < \infty$. A map $f : A \to Y$ is called a $C$-bilipschitz embedding if there exists $r > 0$ such that
\[
\forall u,v \in A \quad r d_A(u,v) \leq d_Y(f(u), f(v)) \leq r Cd_A(u,v). \quad (1)
\]
A bilipschitz embedding is an embedding which is $C$-bilipschitz for some $1 \leq C < \infty$. The smallest constant $C$ for which there exist $r > 0$ such that (1) is satisfied is called the distortion of $f$.

(iii) Let \( \{M_n\}_{n=1}^\infty \) and \( \{R_n\}_{n=1}^\infty \) be two sequences of metric spaces. We say that \( \{M_n\}_{n=1}^\infty \) admits uniformly bilipschitz embeddings into \( \{R_n\}_{n=1}^\infty \) if for each $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ and a bilipschitz map $f_n : M_n \to R_{m(n)}$ such that the distortions of \( \{f_n\} \) are uniformly bounded.

Remark 1.5. It is not difficult to see that a sequence of metric spaces which admits uniformly bilipschitz embeddings into a sequence of uniformly doubling spaces is itself uniformly doubling. Together with the mentioned above fact that the graphs \( \{L_n\}_{n=0}^\infty \) are uniformly doubling while \( \{D_n\}_{n=0}^\infty \) are not, this observation implies that \( \{D_n\}_{n=0}^\infty \) does not admit uniformly bilipschitz embeddings into \( \{L_n\}_{n=0}^\infty \).

The present paper aims to prove the non-embeddability in the opposite direction. This has been achieved by the next theorem, which constitutes the main result of this work.

Theorem 1.6. The sequence of Laakso graphs does not admit uniformly bilipschitz embeddings into the sequence of diamond graphs.

We also observe in Example 3.2 that there is a Laakso-type family of graphs which embed isometrically into diamond graphs. The main motivation for the study undertaken in this paper was to find out whether subspace-hereditary bilipschitz-invariant properties pass from diamond graphs to Laakso graphs, and so one does not have to prove results for them separately, as it was done in [6]. The main conclusion is that in general it is not the case, but it is the case for properties which can be shown to hold for all Laakso-type spaces simultaneously. Another approach to proving results for diamond and Laakso graphs simultaneously is to prove results for classes of graphs which include both diamond and Laakso graphs, possibly in some cases this can be done by developing quantitative versions of results of [15] on thick families of geodesics.

We refer to [1] for graph-theoretical terminology and to [13] for terminology of the theory of metric embeddings.
2 Proof of the main result

Since, clearly, the validity of Theorem 1.6 does not depend on whether weighted or unweighted version of \( \{D_n\} \) and \( \{L_n\} \) is considered, our attention in this section is restricted to unweighted versions of \( \{D_n\} \) and \( \{L_n\} \). The vertex set of a graph \( G \) will be denoted by \( V(G) \), and the edge set by \( E(G) \).

To prove Theorem 1.6, it suffices to show that, for any \( k \in \mathbb{N} \), no matter how the numbers \( m(n) \in \mathbb{N} \) and \( p(n) \in \mathbb{N} \cup \{0\} \) are chosen, it is impossible to find maps \( F_n : V(L_n) \to V(D_{m(n)}) \) such that

\[
\forall n \forall u, v \in V(L_n) \quad 4^{p(n)}d_{L_n}(u, v) \leq d_{D_{m(n)}}(F_n(u), F_n(v)) \leq 4^k \cdot 4^{p(n)}d_{L_n}(u, v). \tag{2}
\]

It should be pointed out that there is no need to consider negative values of \( p(n) \). Indeed, in such cases, one may replace \( D_{m(n)} \) by \( D_{m(n)−2p(n)} \) and use the natural map of \( D_{m(n)} \) into \( D_{m(n)−2p(n)} \), which multiplies all distances by \( 4^{-p(n)} \).

To show the impossibility of finding \( \{F_n\} \) satisfying the condition above assume that such maps exist. Observe that \( L_n \) \((n \geq 1)\) contains \( 4^h \)-cycles isometrically for all \( h \in \{1, \ldots, n\} \). Some families of such cycles, labelled by quaternary trees \( Q_a \) with \( a \in \mathbb{N} \) generations, will be used in the forthcoming reasonings.

**Definition 2.1.** The quaternary tree \( Q_a \) of depth \( a \) is defined as the graph whose vertex set \( V(Q_a) \) is the set of all sequences of length \( \leq a \) with entries \( \{0, 1, 2, 3\} \), including the empty sequence \( \emptyset \); and the edge-set \( E(Q_a) \) is defined by the following rule: two sequences are joined by an edge if and only if one of them is obtained from the other by adding an element at the right end.

For any \( s, t \in \{1, \ldots, n\}, s > t \), we can find a set \( C_{s,t} \) of cycles in \( L_n \) labelled by \( Q_{s−t} \) and satisfying the next two conditions:

- The cycle \( c_\emptyset \) corresponding to \( \emptyset \) has length \( 4^s \) and common vertices with a cycle of length \( 4^n \) in \( L_n \). Note that for \( n > 1 \) there are many such cycles, we pick one of them.

- Let \( \tau \) be a \( \{0, 1, 2, 3\} \)-sequence with \( m \) entries (where \( m < s − t \)), for which we have already defined the corresponding cycle \( c_\tau \) of length \( 4^{s−m} \). Then cycles \( c_\tau,0, c_{\tau,1}, c_{\tau,2} \), and \( c_{\tau,3} \) corresponding to sequences \( \{\tau, 0\}, \{\tau, 1\}, \{\tau, 2\}, \{\tau, 3\} \in Q_{s−t} \) are cycles of length \( 4^{s−m−1} \) such that no pair of them has a common vertex, but each of them has common vertices with \( c_\tau \).

Here, \( L_n \) and \( D_n \) are regarded not only as combinatorial graphs, but also as 1-dimensional simplicial complexes according to the following standard procedure: each edge is identified with a line segment of length 1 and the distance is the shortest path distance (see [17, pp. 82–83]). There is a natural way to extend the maps \( F_n \) to these simplicial complexes in such a manner that the Lipschitz constant of \( F_n \)
does not change: pick, for each edge \( uv \in E(L_n) \), one of the shortest paths between \( F_n(u) \) and \( F_n(v) \) and map the edge into the path so that, for any \( t \in uv \), one has:

\[
d_{D_{m(n)}}(F_n(u), F_n(t)) = d_{L_n}(u, t) \cdot \frac{d_{D_{m(n)}}(F_n(u), F_n(v))}{d_{L_n}(u, v)}.
\]

Select a cycle of length 4 in \( L_n \) and denote it \( C_{4h} \). The assumption on \( F_n \) implies that the images of consecutive vertices of \( C_{4h} \) under \( F_n \) can be joined by the shortest paths of lengths between \( 4p(n) \) and \( 4p(n) + k \). As a result, one obtains a closed walk of length between \( 4p(n) + h \) and \( 4p(n) + k + h \). It can be shown by example that this closed walk does not have to be a cycle as it can have additional self-intersections. However, as it will be proved in Lemma 2.4, for \( h \) which is substantially larger than \( k \), - in the sense described below - the image \( F_n C_{4h} \) has to contain a large cycle in \( D_{m(n)} \).

The following definitions are used in the sequel. Some of them were introduced in [6] and [14].

**Definition 2.2.** (i) A subdiamond of \( D_n \) is a subgraph which evolved from an edge of some \( D_k \) for \( 0 \leq k \leq n \). Similar to the notions of the top and bottom of \( D_n \) introduced above, we define the top (bottom) of a subdiamond \( S \) as its vertex which is the closest to the top (bottom) of \( D_n \) among vertices of \( S \).

(ii) When a subdiamond \( S \) evolves from an edge \( uv \), in the first step we obtain a quadrangle \( u, a, v, b \). The vertex of \( S \) corresponding to \( a \) is called the leftmost, while the vertex of \( S \) corresponding to \( b \) is called the rightmost vertex of \( S \). Obviously, a choice of left and right here is arbitrary, but sketching a diamond in the plane - recall that all of \( \{D_n\} \) and \( \{L_n\} \) are planar graphs - one can follow our choices. The distance between the top and the bottom of a subdiamond is called the height of the subdiamond.

(iii) By a principal cycle in a subdiamond \( S \) of \( D_n \) we mean a cycle which consists of two parts: a path from top to bottom which passes through the leftmost vertex and a path from bottom to top which passes through the rightmost vertex. A principal cycle is considered not only as a graph-theoretical cycle, but also as a 1-dimensional simplicial complex homeomorphic to a circle.

The next lemma reveals a simple property of cycles in diamonds.

**Lemma 2.3.** Each cycle \( C \) in \( D_n \) is a principal cycle in one of subdiamonds of \( D_n \). Each principal cycle in a subdiamond \( S \) of height \( 2^t \) has length \( 2^{t+1} \).

**Proof.** Let \( S \) be the smallest subdiamond in \( D_n \) containing the cycle \( C \). Let \( 2^t \) be the height of \( S \) and let \( S_1, S_2, S_3, \) and \( S_4 \) be the subdiamonds of \( S \) of height \( 2^{t-1} \). See Figure 1. Since \( C \) is not contained in any of \( S_1, S_2, S_3, S_4 \), it has common edges with at least two of them, say, \( S_1 \) and \( S_2 \). It is clear that either the pair top-bottom of \( S \) or the pair leftmost-rightmost vertices separates \( S_1 \) from \( S_2 \) in \( C \). Therefore, both vertices of the corresponding pair should be in the cycle. In the top-bottom case, there should be two paths joining top and bottom. Since the leftmost-rightmost vertices form a cut, one of the paths should go through the leftmost vertex, and the
other through the rightmost vertex. To show that each of them is of length $2^t$, we use the induction on $t$.

For $t = 1$ the situation is clear (sketch $D_1$). Assume that the statement holds for $t = k$ and consider the case $t = k + 1$. Then the path goes either through the leftmost or the rightmost vertex. In any event, it goes from top to bottom of each of the subdiamonds on the corresponding side. Using the induction hypothesis, one obtains the desired result.

The leftmost-rightmost case is treated likewise.

Lemma 2.4. The subgraph $M$ in $D_{m(n)}$ spanned by the closed walk $F_n C_{4h}$ contains a cycle of length $\ell$ satisfying

$$\ell \geq 4^{p(n)} \left( \frac{4h}{3} - 1 - 4^k \right).$$

Proof. Assume the contrary, that is, let the longest cycle contained in $F_n C_{4h}$ be of length $< 4^{p(n)} \left( \frac{4h}{3} - 1 - 4^k \right)$

Denote by $\nu$ the largest integer satisfying

$$2^{\nu} < 4^{p(n)} \left( \frac{4h}{3} - 1 - 4^k \right).$$

We collapse all subdiamonds which have principal cycles contained in $F_n C_{4h}$. By this we mean that, for each subdiamond $S$ of height $2^\mu$ with $\mu \leq \nu$, such that one of its principal cycles is contained in $F_n C_{4h}$, but $S$ is not (properly) contained in a larger subdiamond whose principal cycle is also contained in $F_n C_{4h}$, we do the following. Replace $S$ by a path of length $2^\mu$ and map all edges of the subdiamond onto edges of the path in such a way that their distances from the top and bottom of the subdiamond are preserved. Denote by $G$ the graph obtained by applying this procedure to $D_{m(n)}$, and by $P$ the corresponding map, viewed both as a map $V(D_{m(n)}) \to V(G)$ and $E(D_{m(n)}) \to E(G)$.

Let us prove that $PF_n C_{4h}$ is a tree. Indeed, it is obvious that $PF_n C_{4h}$ cannot contain a cycle $C$ of the form $PC$, where $PC$ is a cycle in $F_n C_{4h}$ since all such cycles $C$ were collapsed. Therefore, any of the preimages $\tilde{C}$ of $C$ in $F_n C_{4h}$ cannot be a cycle. This implies that $C$ contains some of the paths obtained as collapsed subdiamonds, and that, for at least one of them, the preimage of this path in $F_n C_{4h}$ cannot be connected. However, this is impossible according to the preceding definitions.

Therefore, by virtue of [17, Proposition 5.1], there are two points $x, y$ in the cycle $C_{4h}$ (regarded as a 1-dimensional simplicial complex) at distance $d_{L_n}(x, y) \geq 4^h/3$, such that $PF_n(x) = PF_n(y)$. Since $2^\nu$ is the diameter of the largest subdiamond which was collapsed and two distinct points in different collapsed subdiamonds are not mapped to the same point, the distance between two preimages of the same point under $P$ is at most $2^\nu$. Thence, $d_{D_{m(n)}}(F_n(x), F_n(y)) \leq 2^\nu$. As points $x$ and $y$
do not have to be vertices, we find the closest vertices of $C_4^h$ to them, say, $v_x$ and $v_y$. Then $d_{L_n}(x, v_x) \leq \frac{1}{2}$ and $d_{L_n}(y, v_y) \leq \frac{1}{2}$, whence

$$d_{D_{m(n)}}(F_n(v_x), F_n(v_y)) \leq 2^\nu + 4^{p(n)+k}.$$ 

On the other hand, inequality $d_{L_n}(x, y) \geq 4^h / 3$ yields:

$$d_{D_{m(n)}}(F_n(v_x), F_n(v_y)) \geq \left(\frac{4^h}{3} - 1\right) 4^{p(n)}.$$ 

Thus, $2^\nu \geq 4^{p(n)} \left(\frac{4^h}{3} - 1 - 4^k\right)$, contrary to (4).

This lemma implies the following claim.

**Claim 2.5.** If $h$ satisfies

$$\left(\frac{4^h}{3} - 1 - 4^k\right) \geq 4^{h-1},$$

then the $F_n$-image of any (isometric) cycle $C_4^h$ of length $4^h$ in $L_n$ contains a cycle whose length in $D_{m(n)}$ is between $4^{p(n)+h-1}$ and $4^{p(n)+h+k}$.

Notice that the uniqueness of such a cycle in $D_{m(n)}$ is not asserted. Denote by $A(C_4^h)$ any of the cycles satisfying the conditions of Claim 2.5 and by $S(C_4^h)$ the subdiamond for which $A(C_4^h)$ is a principal cycle.

Now, consider the collection $C_{s,t}$ of cycles in $L_n$ introduced after Definition 2.1 and labelled by $Q_{s-t}$. We pick $n > s > t$ in such a way that $h = t$ satisfies (5), and, in addition, the three inequalities below hold:

$$n > s + k,$$  

$$s - t > 2(k + 1),$$  

$$t > 10(k + 1).$$

At this stage, the question arises: Given a cycle $A(c_\tau)$ with sequence $\tau$ of length $m < s - t$, what can be said about the cycles $A(c_\tau,0)$, $A(c_\tau,1)$, $A(c_\tau,2)$, $A(c_\tau,3)$ and the corresponding subdiamonds $S(c_\tau,0)$, $S(c_\tau,1)$, $S(c_\tau,2)$, and $S(c_\tau,3)$? The writing $\tau,0$ means the sequence obtained from $\tau$ by adding 0 at the right end.

Lemmas 2.6, 2.9, and 2.10 below provide an answer to this question.

**Lemma 2.6.** The subdiamonds $S(c_\tau,0), S(c_\tau,1), S(c_\tau,2),$ and $S(c_\tau,3)$ are contained in $S(c_\tau)$, but they contain neither the top nor the bottom of $S(c_\tau)$. 

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Proof. Since $\text{diam}(L_n) = 4^n$, we have $\text{diam}(F_n L_n) \geq 4^{p(n)+n}$. On the other hand, the condition $\text{diam}(c_r) = \frac{1}{2} 4^{s-m}$ implies $\text{diam} S(c_r) = \text{diam} A(c_r) \leq \frac{1}{2} 4^{s-m+p(n)+k} \leq \frac{1}{2} 4^{s+m+p(n)+k}$, where the equality reflects an easy property of principal cycles. Meanwhile, inequality (6) implies that $F_n L_n$ cannot be contained in $S(c_r)$.

Notice that the deletion of $\{v_t, v_b\}$, where $v_t$ is the top and $v_b$ is the bottom of $S(c_r)$ splits the diamond $D_{m(n)}$ into three pieces: the part containing the rightmost vertex of $S(c_r)$, the part containing the leftmost vertex of $S(c_r)$, and the complement of $S(c_r)$. Using the previous paragraph and the definition of $S(c_r)$, one concludes that each of these parts has nonempty preimages under $F^{-1}_n$. Hence, preimages of $v_t$ and $v_b$ split $L_n$ into at least three connected components: the component containing one of the preimages of the leftmost vertex, the component containing one of the preimages of the rightmost vertex, and the component containing a vertex of $L_n$ whose $F_n$-image is not contained in $S(c_r)$.

It can be observed that the subsets $F^{-1}_n(v_t)$ and $F^{-1}_n(v_b)$ in $L_n$ have diameters $\leq 4^k + 1$, and the same is also true for the preimage of any point of $F_n L_n$. Indeed, assume the contrary, that is, let $F_n(x) = F_n(y)$ and $d_{L_n}(x, y) > 4^k + 1$. Let $v_x, v_y \in V(L_n)$ be the nearest to $x$ and $y$ vertices. Then $d_{L_n}(v_x, v_y) > 4^k$, but $d_{D_{m(n)}}(F_n(v_x), F_n(v_y)) \leq 4^{p(n)+k}$. This, however, contradicts (2).

To show that the preimages of $v_t$ and $v_b$ cannot intersect any of $c_{r,0}, c_{r,1}, c_{r,2}$, and $c_{r,3}$, some more information on the action of the map $F_n$ on $c_r$ is required.

The cycle $c_r$ has length $4^{s-m}$ and $s - m > t$. By Claim 2.5, the length of $A(c_r)$ is at least $4^{p(n)+s-m-1} \geq 4^{p(n)+t}$.

Let the cycle $A(c_r)$ be divided into pieces of length $4^{p(n)+k}$. By Lemma 2.3 the length of every cycle in $D_{m(n)}$ is a power of 2. Since by (8), $4^{p(n)+t} > 4^{p(n)+k}$, the length of $A(c_r)$ is divisible by $4^{p(n)+k}$. There are at least $4^{s-m-k-1}$ pieces. Denote the number of pieces by $R$ and the endpoints of the pieces by $u_r, r \in \{0, \ldots, R-1\}$. It is easy to see that $R$ is even. For each $r \in \{0, \ldots, R-1\}$, pick one of the preimages of $u_r$ in $c_r$, and denote it by $a_r$. We may and shall assume that both $v_t$ and $v_b$ are among $\{u_r\}$. Inequality (2) implies that

$$\forall r \in \{0, 1, \ldots, R-1\} \forall q \in \{1, 2, \ldots, R/2\}, \quad d_{L_n}(a_r, a_r+q) \in [q, q \cdot 4^k], \quad (9)$$

where $r + q$ is understood modulo $R$.

One can show by example that the sequence $\{a_r\}$ does not have to be located on $c_r$ in a cyclic order. However, it will be shown that if we replace $4^{p(n)+k}$ in the definition of $\{u_r\}$ by $4^{p(n)+2k}$, the preimages $\{b_r\}$ of the resulting sequence $\{v_r\}$ are located on $c_r$ in a cyclic order.

There is a well-defined notion of an opposite point $\hat{w}$ of a point $w$ in $c_r$. Together $w$ and $\hat{w}$ split $c_r$ into two parts. Points $u$ and $v$ of the cycle $c_r$ are said to be on the same side of $w$ if they belong to the same part. Otherwise we say that $u$ and $v$ are on different sides of $w$.

Lemma 2.7. Any two of the points $a_{4^k}, a_{4^k+1}, a_{4^k+2}, \ldots, a_q$ with $q = \frac{1}{2} 4^{9k}$ are on the same side of $a_0$. 

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Proof. Clearly, it is enough to prove that \( a_{q_1} \) and \( a_{q_2} \) are on the same side of \( a_0 \) if \( q_1 \) and \( q_2 \) are two consecutive elements of \( \{ 4^k, 4^k + 1, 4^k + 2, \ldots, \frac{3}{2} 4^{0k} \} \).

Consider two consecutive elements \( q_1 \) and \( q_2 \) of \( \{ 4^k, 4^k + 1, 4^k + 2, \ldots, \frac{1}{2} 4^{0k} \} \) and assume that \( a_{q_1} \) and \( a_{q_2} \) are on different sides of \( a_0 \). We obtain:

\[
d_{L_n}(a_{q_1}, a_{q_2}) = \min\{d_{L_n}(a_{q_1}, a_0) + d_{L_n}(a_0, a_{q_2}), 4^{s-m} - (d_{L_n}(a_{q_1}, a_0) + d_{L_n}(a_0, a_{q_2}))\}
\]

\[\geq \min\{q_1 + q_2, 4^{s-m} - (q_1 + q_2)4^k\} \geq q_1 + q_2 \geq 2 \cdot 4^k + 1,
\]

while the distance between \( u_{q_1} \) and \( u_{q_2} \) is \( 4^{p(n)+k} \). This contradicts (2). \( \square \)

Now, let \( v_r = u_{r,4^k}, r = 1, 2, \ldots \), and let \( b_r = a_{r,4^k} \) be the preimages of \( v_r \).

**Lemma 2.8.** The sequence \( \{b_r\} \) is placed on the cycle \( c_r \) in its natural order.

Proof. Lemma 2.7 implies that for every \( r \), the points \( b_{r+1} \) and \( b_{r+2} \) are on the same side of \( b_r \) (addition here and below is modulo the cardinality of \( \{b_r\} \)). If each such triple \( b_r, b_{r+1}, b_{r+2} \) is in the natural order on the cycle (by this we mean that the shortest path between \( b_r \) and \( b_{r+2} \) on \( c_r \) passes through \( b_{r+1} \)), we are done.

It remains to consider the case where, for some \( r \), the order is \( b_r, b_{r+2}, b_{r+1} \) (as before, by this we mean that the shortest path between \( b_r \) and \( b_{r+1} \) on \( c_r \) passes through \( b_{r+2} \)). Apply Lemma 2.7 to two pairs: (1) \( b_r = a_{r,4^k} \) (playing the role of \( a_0 \)) and \( b_{r+2} = a_{(r+2),4^k} \) and (2) \( b_{r+1} = a_{(r+1),4^k} \) (playing the role of \( a_0 \)) and \( b_r + 2 = a_{(r+2),4^k} \). As a result we conclude that points \( a_{(r+2)4^k+i}, i = 1, 2, \ldots, 4^{8k} \) should be between \( b_r \) and \( b_{r+1} \). By (9), this is impossible. \( \square \)

It is easy to see that one may assume that \( v_t \) and \( v_b \) are among \( \{v_r\} \), and thence, the adopted notation is consistent. With some abuse of notation, assume that \( t \) and \( b \) are not only the abbreviations for ‘top’ and ‘bottom’, but also integers. We denote the corresponding elements of \( \{b_r\} \) by \( b_t \) and \( b_b \).

Observe that the cycle \( c_T \) contains two vertices whose removal disconnects the remaining vertices of \( c_T \) from the large-diameter part of \( L_n \). Denote these vertices by \( c_T \) and \( c_B \), respectively. Consider pieces of \( c_T \) between \( b_{t-2} \) and \( b_{t+2} \) and also between \( b_{b-2} \) and \( b_{b+2} \); we claim that both \( c_T \) and \( c_B \) belong to one of these pieces of \( c_T \). Note that, using the triangle inequality it is easy to verify that \( c_T \) and \( c_B \) are in different pieces.

We prove that the assumption that \( c_T \) is in none of the mentioned pieces leads to a contradiction. Assume that \( d_{L_n}(c_T, b_{t-2}) \) does not exceed any of the \( d_{L_n}(c_T, b_{t+2}) \), \( d_{L_n}(c_T, b_{b-2}) \), \( d_{L_n}(c_T, b_{b+2}) \), all other cases are similar.

The proven above estimate \( \text{diam}(F^{-1}_n(v_t)) \leq 4^k + 1 \) implies that the set \( F^{-1}_n(v_t) \) is in the \( (4^k + 1) \)-neighborhood of \( b_t \), and therefore the assumption just made implies, by (9), that \( F^{-1}_n(v_t) \) does not intersect the path joining \( c_T \) and \( b_{t-2} \). For the same reason \( F^{-1}_n(v_b) \) is in the \( (4^k + 1) \)-neighborhood of \( b_b \), and therefore does not intersect the path joining \( c_T \) and \( b_{t-2} \). Also, since \( c_T \) is in none of the mentioned pieces, we conclude also that \( F^{-1}_n(v_t) \) and \( F^{-1}_n(v_b) \) cannot intersect the path which joins \( c_T \).
with the large component of the graph obtained after the removal of $F_n^{-1}(v_t)$ and $F_n^{-1}(v_b)$ from $L_n$.

Therefore, after the removal of $F_n^{-1}(v_t)$ and $F_n^{-1}(v_b)$ from $L_n$, the vertex $c_T$ will end up in the same component of $L_n$ as $b_{t-2}$, and the diameter of this component is the same as the diameter of $L_n$. On the other hand, its image is the containing $v_{t-2}$ component of the diamond $D_{m(n)}$ with vertices $\{v_t, v_b\}$ deleted, so it is contained in $S(c_T)$. Combining this conclusion with the argument used at the very beginning of the proof of Lemma 2.6 we get a contradiction. The proof for $c_B$ is similar.

Observe that the construction of Laakso graphs is such that the distance between vertices $c_T$ and $c_B$ and any of the cycles intersecting $c_T$ whose length is a quarter of the length of $c_T$, so it is $4^{s-m-2} \geq 4^{t-1}$. Hence the distance between $c_T$ or $c_B$ and any of $c_T,0, c_T,1, c_T,2$, and $c_T,3$ is $4^{s-m-2} \geq 4^{t-1}$. On the other hand, the distance between $b_{t-2}$ and $b_{t+2}$ is at most $4^{2k}$ (and the same holds for $b_{t-2}$ and $b_{t+2}$). Applying the proved above statement about $c_T$ and $c_B$ and inequality (8), we infer that $F_n$-images $c_T,0, c_T,1, c_T,2$, and $c_T,3$ (considered as 1-dimensional simplicial complexes) do not contain $v_t$ and $v_b$. On the other hand, it is easy to see, for example from Lemma 2.8, that these images intersect $S(c_T)$. The conclusion of Lemma 2.6 follows.

Lemma 2.9. The cycles $A(c_T,0), A(c_T,1), A(c_T,2)$, and $A(c_T,3)$ are disjoint.

Proof. Here, it will be shown that the existence of common vertices contradicts the bilipschitz condition (2). In fact, if there are points $z \in c_T, i \in \{0,1,2,3\}$, for which $i \neq j$ and their images coincide, then there exist vertices $v_z \in c_T, i$ and $v_w \in c_T, j$ in the cycles with $d_{L_n}(z, v_z) \leq \frac{1}{2}$ and $d_{L_n}(w, v_w) \leq \frac{1}{2}$. Therefore, $d_{D_{m(n)}}(F_n(v_z), F_n(v_w)) \leq 4^{p(n)+k}$. On the other hand, the definition of $c_T, i$, $c_T, j$ yields $d_{L_n}(v_z, v_w) \geq \frac{1}{2} 4^{t}$, and thus $d_{D_{m(n)}}(F_n(v_z), F_n(v_w)) \geq \frac{1}{2} 4^{p(n)+t}$. This contradicts condition (8).

Lemma 2.10. Two of the subdiamonds $S(c_T,0), S(c_T,1), S(c_T,2), S(c_T,3)$ have diameters at least 8 times smaller than the diameter of $S(c_T)$.

Proof. Lemma 2.6 excludes all subdiamonds of $S(c_T)$ of diameter $\frac{1}{4} \text{diam}(S(c_T))$ as candidates for $S(c_T,0), S(c_T,1), S(c_T,2), S(c_T,3)$. What is more, Lemma 2.6 excludes 8 out of 16 subdiamonds of $S(c_T)$ of diameter $\frac{1}{4} \text{diam}(S(c_T))$, as can be viewed by sketching $D_3$. Consequently, we are left with two 4-tuples of subdiamonds of $S(c_T)$ of diameter $\frac{1}{4} \text{diam}(S(c_T))$. Subdiamonds in each 4-tuple have a common vertex. Therefore, by Lemma 2.9 only two of these subdiamonds can serve as one of $S(c_T,0), S(c_T,1), S(c_T,2), S(c_T,3)$. The conclusion follows.

Proof of Theorem 1.6. Lemma 2.10 implies that the diameter of at least one of the subdiamonds $S(c_T)$, where sequence $\tau$ comprises $s-t$ elements, does not exceed

$$\text{diam}(S(c_T)) \cdot 8^{-t} \leq 4^{p(n)+k} \cdot \left(\frac{1}{2} 4^t\right) \cdot 8^{-t}.$$
On the other hand, the assumption stating that \( h = t \) satisfies (5) implies the inequality \( \text{diam}(S(c_\tau)) \geq \frac{1}{2} 4^{p(n) + t - 1} \) leading to

\[
\frac{1}{2} 4^{p(n) + t - 1} \leq 4^{p(n) + k} \cdot \left( \frac{1}{2} 4^s \right) \cdot 8^{-(s-t)}
\]
or \( 2^{s-t} \leq 4^{k+1} \). This contradicts (7).

**Remark 2.11.** It is easy to check that the reasoning above implies that there exists \( C \geq 1 \) - our argument shows that any \( C \geq 26 \) works - such that for any \( k \in \mathbb{N} \) and any \( n \geq Ck \) the distortion of any embedding of \( L_n \) into any of the diamond graphs is at least \( 4^k \).

3 Consequences and discussion

As an immediate outcome of Theorem 1.6 one obtains the following statement:

**Corollary 3.1.** \( L_\omega \) does not admit a bilipschitz embedding into \( D_\omega \).

**Proof.** Assume the contrary. Since \( L_\omega \) contains isometric copies of (weighted) Laakso graphs \( \{L_n\}_{n=0}^\infty \), this would imply that \( \{L_n\}_{n=0}^\infty \) admits uniformly bilipschitz embeddings into \( D_\omega \). Further, \( D_\omega \) is the union of \( \{D_n\}_{n=0}^\infty \) implying that \( \{L_n\}_{n=0}^\infty \) admits uniformly bilipschitz embeddings into \( \{D_n\}_{n=0}^\infty \). This, however, contradicts Theorem 1.6.

Corollary 3.1 rules out the approach to the proof of nonembeddability of \( D_\omega \) into a Banach space with the Radon-Nikodým property (RNP) ([12, Corollary 3.3]) by an immediate application of the corresponding result for the Laakso space, which can be found in [3, Corollary 1.7] and [12, Theorem 3.6].

However, as it can be seen from the proof above, the reason for the non-embeddability of Laakso graphs into diamond graphs is rather specific: an incompatible choice of the parameters. One can easily adjust Laakso graphs and get a family of Laakso-type graphs which are isometrically embeddable into diamonds, as illustrated by Example 3.2 below. An analogue of the RNP-nonembeddability result for such Laakso-type spaces immediately implies the non-embeddability of the infinite diamond \( D_\omega \) into Banach spaces with the RNP.

**Example 3.2.** Repeat the Laakso construction with \( L_1 \) replaced by the graph \( M_1 \) shown in Figure 3. Denote the obtained graphs \( \{M_n\}_{n=0}^\infty \). It is easy to verify that \( M_1 \) embeds isometrically into \( D_3 \). Finally, it can be proved by induction on \( n \) that \( M_n \) embeds isometrically into \( D_{3n} \) for all \( n \in \mathbb{N} \).

Another way of proving the result simultaneously for \( D_\omega \) and \( L_\omega \) was found in [16]. It is based on the notion of *thick family of geodesics*. The result of Mendel and Naor [10] on the lack of Markov convexity in the Laakso space also can be generalized to the case of spaces with thick families of geodesics, which includes the infinite diamond \( D_\omega \). See [15] for this and related issues.
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