Minimum congestion spanning trees in bipartite and random graphs

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Abstract. The first problem considered in this paper: is it possible to find upper estimates for the spanning tree congestion for bipartite graphs which are better than for general graphs? It is proved that there exists a bipartite version of the known graph with spanning tree congestion of order $n^{\frac{3}{2}}$, where $n$ is the number of vertices. The second problem is to estimate spanning tree congestion of random graphs. It is proved that the standard model of random graphs cannot be used to find graphs whose spanning tree congestion has order greater than $n^{\frac{3}{2}}$.

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1 Introduction

In this paper we consider finite simple graphs. Our graph-theoretic terminology follows [1]. For a graph $G$ by $V_G$ and $E_G$ we denote its vertex set and its edge set, respectively.

Let $G$ be a graph and let $T$ be a spanning tree in $G$ (that is, $T$ is both a tree and a spanning subgraph of $G$). For each edge $e$ of $T$ let $A_e$ and $B_e$ be the vertex sets of the components of $T \setminus e$. By $e_G(A_e, B_e)$ we denote the number of edges with one end vertex in $A_e$ and the other end vertex in $B_e$. We define the edge congestion of $G$ in $T$ by

$$ec(G : T) = \max_{e \in E_T} e_G(A_e, B_e).$$

The name comes from the following analogy. Imagine that edges of $G$ are roads, and that edges of $T$ are those roads which are cleaned from snow after snowstorms. For each road $g \in E_G$ there exists a unique path $P_g$ in $T$ joining the end vertices of $g$, we call such path a detour, even in the case when $P_g = g$. For an edge $h$ of $T$ it is quite natural to define the congestion $c(h)$ as the number of times $h$ is used in different detours $\{P_g\}_g \in E_G$. Then

$$ec(G : T) = \max_{h \in E_T} c(h).$$

It is clear that for applications it is interesting to find a spanning tree which minimizes the congestion.

We define the spanning tree congestion of $G$ by

$$s(G) = \min\{ec(G : T) : T \text{ is a spanning tree of } G\}. \quad (1)$$

Each spanning tree $T$ in $G$ satisfying $ec(G : T) = s(G)$ is called a minimum congestion spanning tree for $G$. The parameters $ec(G : T)$ and $s(G)$ were introduced and studied in [2]. These parameters are of interest in the study of Banach-space-theoretical properties of Sobolev spaces on graphs, see [3, Section 3.5.1]. An additional motivation for this study is that minimum congestion spanning trees can be considered as ‘congestional’ analogues of the well-known shortest (or minimal) spanning trees. See [4] for an account on shortest spanning trees, and on other minimality properties of spanning trees.

One of the natural questions about $s(G)$ is to find the maximal possible value $\mu(n)$ of $s(G)$ for graphs $G$ with $n$ vertices. The rate of growth of the function $\mu(n)$ was studied in Section 3 of [2], where it was shown that $c_1 n^{3/2} \leq \mu(n) \leq c_2 n^2$ for some absolute constants $c_1$ and $c_2$, and some estimates for $c_1$ and $c_2$ were found. The first problem considered in this paper is: is it possible to find upper estimates of $s(G)$ for bipartite graphs which are better than for general graphs? In connection with this problem we prove that there exists a bipartite version of the graph with spanning tree congestion of order $n^{3/2}$ from [2, Theorem 2], but it has to have a larger radius than the example from [2]. In Section 3 we study spanning tree congestion of random graphs. As is well known random graphs can be used to find examples with close-to-extremal behavior for many problems. We prove that the standard model of random graphs cannot be used to improve the exponent $3/2$ in the lower estimate for $\mu(n)$. 

2
2 Minimum congestion spanning trees in bipartite graphs

In [2, Theorem 2] it was shown that there exist graphs \( G \) with \( n \) vertices and \( s(G) \geq cn^{3/2} \), where \( c > 0 \) is an absolute constant. Our first purpose is to show that (with a bit worse estimate for \( c \)) the same result remains true for bipartite graphs. We refer to [5] for a systematic exposition of the theory of bipartite graphs.

We use the following way of getting a bipartite graph from an arbitrary graph, we call it *duplication*. Let \( H \) be a graph with vertex set \( V_H \). By duplication of the graph \( H \) we mean the bipartite graph \( \tilde{H} \) defined in the following way. We replace each vertex \( v \in V_H \) by a pair of vertices \( u_v, w_v \), and let \( V_{\tilde{H}} = \{ u_v, w_v : v \in V_H \} \). The vertex \( u_v \) is adjacent with \( w_x \) in \( \tilde{H} \) if an only if \( x \in V_H \) is either \( v \) or a vertex adjacent to \( v \). It is clear that \( \tilde{H} \) is a bipartite graph and that the sets \( \{ u_v \}_{v \in V_H} \); \( \{ w_v \}_{v \in V_H} \) form a bipartition.

We are going to show that the spanning tree congestion of the duplication of the graph \( G \) considered in [2, Theorem 2] satisfies the same estimate below as \( s(G) \). Recall the definition of \( G \):

Let \( n \) be of the form \( n = 3k - 2\sqrt{k} \), where \( k \) is such that \( \sqrt{k} \) is an integer. We also assume that \( k > 4 \). The graph \( G \) has \( n \) vertices and is defined in the following way. We represent \( V_G \) as a union of 3 sets, \( V_G = C_1 \cup C_2 \cup C_3 \), where \( |C_1| = |C_2| = |C_3| = k \), \( |C_1 \cap C_2| = |C_2 \cap C_3| = \sqrt{k} \), and \( C_1 \cap C_3 = \emptyset \). The edge set of \( G \) is defined in the following way: vertices \( v \) and \( u \) in \( G \) are adjacent if and only if \( u, v \in C_i \) for some \( i \in \{1, 2, 3\} \).

**Theorem 1** Denote by \( \tilde{G} \) the duplication of the graph \( G \). Then \( \tilde{G} \) is a bipartite graph with \( 2n = 6k - 4\sqrt{k} \) vertices and \( s(\tilde{G}) \geq \frac{1}{2} k^{3/2} \).

**Remark.** Our proof follows the lines of the proof of Theorem 2 in [2], with modifications needed because of the duplication.

**Proof.** Working with trees it will be convenient to use related definitions and observations that are going back to C. Jordan [6], see [7, pp. 35–36]. Let \( u \) be a vertex of a tree \( T \). If we delete all edges incident with \( u \) from \( T \), we get a forest. The maximal number of vertices in components of the forest is called the *weight of \( T \) at \( u \).* A vertex \( v \) of \( T \) is called a *centroid vertex* if the weight of \( T \) at \( v \) is minimal. Each tree has one or two centroid vertices.

Let \( T \) be a spanning tree in \( \tilde{G} \) satisfying \( s(\tilde{G}) = ec(\tilde{G} : T) \). Let \( c \) be a centroid of \( T \), if \( T \) has two centroids we choose one of them, and fix the notation \( c \) for it. Denote the sets \( \{ v \}_{v \in C_i} \) and \( \{ u \}_{v \in C_i} \) by \( D_i \) and \( E_i \), respectively, let \( \tilde{C}_i = D_i \cup E_i \). Since the sets \( \tilde{C}_1 \) and \( \tilde{C}_3 \) are disjoint, the centroid \( c \) does not belong to at least one of them. Without loss of generality we assume \( c \notin \tilde{C}_1 \). The edges incident with \( c \) in \( T \) will be called *central edges*. If we remove all central edges from \( T \), we get a forest. Let \( V_1, \ldots, V_l \) be vertex sets of connected components of the forest, except the component whose only vertex is \( c \).
There is a natural bijective correspondence between the set of central edges and the set \{V_1, \ldots, V_t\}. Let us denote the central edge corresponding to \(V_t\) by \(e_t\).

Observe that \(\tilde{C}_1\) cannot intersect more than \(2\sqrt{k}\) of the sets \(V_1, \ldots, V_t\). In fact, only \(2\sqrt{k}\) vertices of \(\tilde{C}_1\) are adjacent to vertices that are not in \(\tilde{C}_1\), so there are only \(2\sqrt{k}\) “entrances” into \(\tilde{C}_1\).

Since \(\tilde{C}_1\) contains \(2k\) vertices, this implies that there exists \(j \in \{1, \ldots, t\}\) such that the intersection \(V_j \cap \tilde{C}_1\) has at least \(\sqrt{k}\) vertices.

Let \(\gamma_1 = |V_j \cap \tilde{C}_1|\). Then \(\gamma_1 = \delta_1 + \epsilon_1\), where \(\delta_1 = |V_j \cap D_1|\) and \(\epsilon_1 = |V_j \cap E_1|\). There are \(\delta_1(k - \epsilon_1) + \epsilon_1(k - \delta_1) = \gamma_1 k - 2\epsilon_1 \delta_1\) edges joining \(V_j \cap \tilde{C}_1\) with \(\tilde{C}_1 \setminus V_j\). Also \(\epsilon_1 \delta_1 \leq \frac{\gamma_1^2}{4}\).

Hence \(e_j\) is used in at least \(\left(k - \frac{\gamma_1}{2}\right)\gamma_1\) detours. Since \(\gamma_1 \geq \sqrt{k}\) and \(k > 4\), then either \(\left(k - \frac{\gamma_1}{2}\right)\gamma_1 \geq \frac{1}{2} k^{3/2}\) or \(k - \frac{\gamma_1}{2} < \frac{1}{2} \sqrt{k}\). In the former case we are done. In the latter case \(\gamma_1 = \epsilon_1 + \delta_1 > 2k - \sqrt{k}\), hence at most \(\sqrt{k}\) of the \(2\sqrt{k}\) elements of the set \(\tilde{C}_1 \setminus \tilde{C}_2\) are not in \(V_j\), and we get \(|\tilde{C}_1 \cap \tilde{C}_2 \cap V_j| > \sqrt{k}\). Therefore \(|\tilde{C}_2 \cap V_j| > \sqrt{k}\). Let \(\gamma_2 = |\tilde{C}_2 \cap V_j|\).

Arguing in the same way as above, there are more than \(\left(k - \frac{\gamma_2}{2}\right)\gamma_2\) edges joining \(V_j \cap \tilde{C}_2\) with \(\tilde{C}_2 \setminus V_j\). Hence \(e_j\) is used in at least \(\left(k - \frac{\gamma_2}{2}\right)\gamma_2\) detours. Since \(\gamma_2 > \sqrt{k}\) and \(k > 4\), then either \(\left(k - \frac{\gamma_2}{2}\right)\gamma_2 \geq \frac{1}{2} k^{3/2}\) or \(k - \frac{\gamma_2}{2} < \frac{1}{2} \sqrt{k}\).

In the former case we are done. In the latter case we get

\[|V_j| > |\tilde{C}_1| + |\tilde{C}_2| - |\tilde{C}_1 \cap \tilde{C}_2| - |\tilde{C}_1 \setminus V_j| - |\tilde{C}_2 \setminus V_j| = 4k - 4\sqrt{k}.
\]

If \(k > 4\), this inequality implies \(|V_j| > |V_{\tilde{G}}|/2\). We get a contradiction with the fact that \(c\) is a centroid. \(\blacksquare\)

It would be interesting to find out for general graphs: how does the duplication affect the parameter \(s(G)\)? In particular, we suggest the following problem.

**Problem 1** Let \(G\) be a connected graph and \(\tilde{G}\) be the graph obtained by duplication of \(G\). Is it true that \(s(\tilde{G}) \geq s(G)\)? (Or, at least, \(s(\tilde{G}) \geq c \cdot s(G)\) for some absolute constant \(c > 0\)?)

Recall (see [1, p. 10]) that the *radius* of a graph \(G\) is \(\min \max d_G(v, w)\), where \(d_G\) is the standard metric on \(G\). Observe that the graph from [2, Theorem 2] described above has radius 2, and its duplication has radius 3. The following result shows that for bipartite graphs of radius 2 the quantity \(s(G)\) cannot be large.

**Theorem 2** Let \(G\) be a bipartite graph of radius 2. Then \(s(G) \leq 3|V_G|\).

**Proof.** Consider a vertex \(v\) in \(G\) satisfying \(d_G(u, v) \leq 2\) for every \(u \in V_G\).
Let $B = \{ u \in V_G : d_G(u, v) = 1 \}$. Let $m = |B|$ and $B = \{ b_1, \ldots, b_m \}$. Let $A = \{ u \in V_G : d_G(u, v) = 2 \}$. We consider a partition $\{ A_i \}_{i=1}^m$ of $A$ satisfying the conditions

- All vertices from $A_i$ are adjacent to $b_i$.

- The sum $\sum_{i,j=1}^m |A_i| - |A_j|$ has the minimal possible value.

Let $T$ be the tree with the following edge set

$$E_T = \{ vb_i : i = 1, \ldots, m \} \cup \{ b_i a : a \in A_i, i = 1, \ldots, m \}. $$

(By $uv$ we denote the edge joining vertices $u$ and $v$.) It is easy to see that $T$ is a spanning tree. Let us estimate $ec(\lambda : T)$. It is easy to see that an edge of the form $b_i a$ is used in $d_a(n - 1)$ detours ($d_a$ is the degree of $a$).

So it remains to estimate the number of detours using an edge of the form $vb_i$. If $U$ and $W$ are sets of vertices of $G$, we use the notation $E_G(U, W)$ for the set of edges with one end vertex in $U$ and the other end vertex in $W$. The edge $vb_i$ is used in detours for

$$D := E_G(b_i) \cup (\bigcup_{j \neq i} E_G(b_j, A_j)) \cup E_G(A_i, b_i).$$

(Since $G$ is bipartite, there are no edges between different vertices from $A$ and no edges between different vertices from $B$.) Let $n = |V_G|$. Then

$$|D| \leq 1 + (n - (m + 1)) + |E_G(A_i, \{ b_j \}_{j \neq i})|. $$

**Observation.** A vertex $b_i$ is not adjacent to any vertices from $A$, if $A_s$ satisfies $|A_s| > |A_i| + 1$.

In fact, otherwise we choose a vertex from $A$, which is adjacent to $b_i$, and move it to $A_i$. We get a new partition $\{ A'_i \}$ satisfying $|A'_s| = |A_s| - 1$, $|A'_i| = |A_i| + 1$, and $|A'_k| = |A_k|$ for $k \notin \{ s, t \}$.

It is easy to see that

$$\sum_{i, j=1}^m |A_i' - A_j'| < \sum_{i, j=1}^m |A_i| - |A_j|. $$

By the observation

$$|E_G(A_i, \{ b_j \}_{j \neq i})| \leq |A_i| \cdot |\{ j : |A_j| \geq |A_i| - 1 \}| \leq |A_i| \frac{n - (m + 1)}{|A_i| - 1} \leq 2(n - (m + 1)), $$

if $|A_i| > 1$. We get an estimate $|E_G(A_i, \{ b_j \}_{j \neq i})| \leq m - 1$ if $|A_i| = 1$. Hence $|D| \leq \max\{ 3n - 3m - 2, n - 1 \}$. Therefore $ec(\lambda : T) < 3n = 3|V_G|$. ■
3 Minimum congestion spanning trees in random graphs

Let \( \mathcal{G}(n,p) \) be the standard probability space of graphs (see [1, p. 217]). We denote the standard probability on this space by \( \mathbb{P}_p \). Our purpose is to show that the spanning tree congestion of graphs in \( \mathcal{G}(n,p) \) is, with high probability, \( O(n^{3/2}) \). Our approaches to showing that the spanning tree congestion is ‘small’ are different in the cases \( p \leq n^{1/2} \) and \( p \geq n^{1/2} \). In the first case the random graph just does not have enough edges to have ‘large’ congestion. In the second case the graph has, with high probability, nice spanning trees (see Theorem 3 for precise meaning of this statement), whose existence implies the desired bound for the congestion.

First we consider the case of ‘small’ \( p \). Let \( A = A_{n,p} \) be the intersection of the following two events in \( \mathcal{G}(n,p) \): \( \{ G_{n,p} \text{ is connected} \} \) and \( \{ G_{n,p} : s(G_{n,p}) \geq n^{3/2} \} \).

Proposition 1 If \( p \leq n^{-1/2} \), then \( \mathbb{P}_p(A) \leq \exp \left( -\frac{3}{16} n^{3/2} \right) \).

Proof. We use the following straightforward estimate \( s(G) \leq |E_G| - |V_G| + 2 \) (see [2, Theorem 1(a)]). Hence \( \mathbb{P}_p(A) \leq \mathbb{P}_p \{ G_{n,p} \text{ has at least } n^{3/2} \text{ edges} \} \). The expected number \( \lambda \) of edges of \( G_{n,p} \) is \( \frac{n(n-1)}{2} p \). If \( p \leq n^{-1/2} \), then \( \lambda < \frac{n^{3}}{2} \). By a Chernoff-type bound (see [8, Formula (2.5), p. 26]),

\[
\mathbb{P}_p \{ G_{n,p} \text{ has at least } n^{3/2} \text{ edges} \} \leq \exp \left( -\frac{(n^{3/2} - \lambda)^2}{2(\lambda + \frac{n^{3/2} - \lambda}{3/2})} \right)
\]

\[
\leq \exp \left( -\frac{(n^{3/2}/2)^2}{2 \cdot \left( \frac{2}{3} n^{3/2} \right)} \right) = \exp \left( -\frac{3}{16} n^{3/2} \right).
\]

Now let \( n^{-1/2} \leq p \leq 1 \) and let \( n \in \mathbb{N} \) be fixed (the discussion below assumes, also, that \( n \) is not very small). Let \( s = \lfloor \frac{n}{s} \rfloor \), and \( d = \left\lceil \log_2 \left( \frac{n-1}{s} \right) \right\rceil \). We assume that \( n \) is large enough so that \( s \geq 2 \). A tree \( T \) will be called a nice tree of depth \( k \), \( k \in \{0, 1, 2, \ldots, d\} \) corresponding to the pair \( (p,n) \) if it has a vertex \( v_0 \) of degree \( s \), at most \( n \) vertices, and the cardinalities of the vertex sets of the \( s \) components of \( T \) obtained after removal of \( v_0 \), we call such components branches, satisfy the following conditions:

- If \( 0 \leq k \leq \left\lfloor \log_2 \left( \frac{n-1}{s} \right) \right\rfloor \), then each of the branches has \( 2^k \) vertices.
- If \( \log_2 \left( \frac{n-1}{s} \right) \) is not an integer, and \( k = \left\lceil \log_2 \left( \frac{n-1}{s} \right) \right\rceil \), then each of the branches has between \( 2^{k-1} \) and \( 2^k \) vertices, and the tree has \( n \) vertices.
Denote by $T_k \in \mathcal{G}(n,p)$ the event of existence of a subgraph which is a nice tree of depth $k$.

**Theorem 3** For each $k \in \{0, \ldots, d\}$ the estimate $\mathbb{P}_p(T_k) \geq (1 - \exp(-c_1 \sqrt{n}))^{k+1} \geq 1 - \exp(-c_2 \sqrt{n})$ holds, where $c_1 > 0$ and $c_2 > 0$ are absolute constants.

**Remark.** To relate this theorem with our study of minimum congestion spanning trees we observe that a nice tree of depth $d$ has the following properties:

- It is a spanning tree.
- If we delete any of its edges, one of the obtained components has at most $2^d \leq 2 \cdot \frac{n-1}{s}$ vertices. Hence the edge is used in at most $2n \cdot \frac{n-1}{s}$ detours.

Therefore these conditions imply (for each graph in which $T$ is a spanning tree) that $\text{ec}(G : T) \leq 2n \cdot \frac{n-1}{s} \leq Cn^{\frac{3}{2}}$.

**Proof of Theorem 3.** We estimate $\mathbb{P}_p(T_k)$ using induction. Observe that $T_0$ is the event of existence of a vertex whose degree is at least $s$. The high probability of this event follows from a Chernoff-type estimate. In fact, the expected degree of a vertex in $p(n-1)$. Hence for a fixed vertex $v_0$ (whose degree we denote by $d_{v_0}$) the probability $\alpha := \mathbb{P}_p\{d_{v_0} \leq \frac{1}{2} p(n-1)\}$ can be estimated using [8, inequality (2.6), p. 26]. We get

$$\alpha \leq \exp \left( - \frac{\left(\frac{1}{2} p(n-1)\right)^2}{2p(n-1)} \right) = \exp \left( - \frac{1}{8} p(n-1) \right) \leq \exp \left( - \frac{n-1}{8\sqrt{n}} \right)$$

and $\mathbb{P}_p(T_0) \geq 1 - \exp \left( - \frac{n-1}{8\sqrt{n}} \right) \geq 1 - \exp(-c_1 \sqrt{n})$.

To estimate $\mathbb{P}_p(T_k)$ for $k > 0$ we use conditional probabilities: $\mathbb{P}_p(T_k) = \mathbb{P}_p(T_{k-1}) \cdot \mathbb{P}_p(T_k | T_{k-1})$. We estimate $\mathbb{P}_p(T_k | T_{k-1})$ using the following lemma. Recall that a set of edges is called a matching if no two of them have a common vertex.

**Lemma 1** Let $A$ and $B$ be disjoint vertex sets satisfying $|A| \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $|B| \geq \left\lceil \frac{n}{2} \right\rceil$. Then the probability that $G_{n,p}$ does not contain a matching $F_A$, such that each edge from $F_A$ has one end vertex in $A$ and one end vertex in $B$, and each vertex from $A$ is incident with exactly one edge from $F_A$, is $\leq \exp(-c_1 \sqrt{n})$ for some absolute constant $c_1 > 0$.

**Proof.** It is clear that the probability is the least if $|A| = |B| = \left\lceil \frac{n}{2} \right\rceil$. In such a case the probability can be estimated using the method from B. Bollobás and A. Thomason [9, pp. 63–65] (see, also, [10, pp. 158–159] and [8, pp. 82–84]). Since in our case $p \geq n^{-\frac{1}{4}}$, we can use more crude and simple estimates. We denote $\left\lfloor \frac{n}{2} \right\rfloor$ by $m$ and write the estimates from [9] with $m$ instead on $n$. We need a version of the following lemma from [9, Lemma 6, p. 64]. As is mentioned in [9], it can be easily derived from Hall’s theorem (which can be found, e.g, in [1, p. 77]).
Lemma 2 ([9]) Let $G$ be a bipartite graph with vertex classes $V_1$ and $V_2$, $|V_1| = |V_2| = m$. Suppose that $G$ does not have a matching $F$, such that each of the vertices is incident with exactly one edge from $F$. Then there is a subset $N \subset V_i$ ($i = 1, 2$) such that

(i) $\Gamma(N) := \{y : y \text{ is adjacent to some } x \in N\}$ has $|N| - 1$ elements.

(ii) $1 \leq |N| \leq (m + 1)/2$.

We consider the bipartite subgraph of $G_{n,p}$ whose vertex classes are $V_1 = A$ and $V_2 = B$ (the edge set of this subgraph consists of all edges of $G_{n,p}$ having one end vertex in $A$ and one end vertex in $B$). Denote by $I_a$ ($a = 1, 2, \ldots, \lfloor(m + 1)/2\rfloor$) the event that there is a set $N \subset V_i$ ($i = 1, 2$), $|N| = a$ satisfying (i) and (ii) for this subgraph. Since we have $2\binom{m}{a}$ choices for $N$ with $|N| = a$ and $\binom{m}{a-1}$ more choices for $\Gamma(N)$, we get

$$\mathbb{P}_p(I_a) \leq 2\binom{m}{a} (m - 1) (1 - p)^{a(m + 1 - a)}$$

Using the elementary estimates $\binom{m}{a} \leq m^a$ and $1 - p < e^{-p}$, we get

$$\mathbb{P}_p(I_a) \leq 2m^a \exp(-pa(m + 1 - a)) \leq 2m^a \exp\left(-p \cdot \frac{m + 1}{2}\right)$$

Let $m_1 = \left\lfloor \frac{m + 1}{2} \right\rfloor$. By Lemma 2 the probability of non-existence of a set $F_A$ satisfying the condition of Lemma 1 can be estimated from above by $\mathbb{P}_p\left(\bigcup_{a=1}^{m_1} I_a\right)$ We have the estimate

$$\mathbb{P}_p\left(\bigcup_{a=1}^{m_1} I_a\right) \leq \sum_{a=1}^{m_1} 2m^a \exp\left(-p \cdot \frac{m + 1}{2}\right) \leq 2m^a \exp\left(-p \cdot \frac{m + 1}{2}\right) \frac{1}{1 - m^2 \exp\left(-p \cdot \frac{m + 1}{2}\right)} \leq \exp(-c_1 n^{1/2})$$

for some absolute constant $c_1$ and sufficiently large $n$. (Of course, we can select the same constant here and in our estimate for $\mathbb{P}_p(T_0)$.)

We continue our proof of the theorem. Suppose that we have proved the estimate for $T_{k-1}$. If a nice tree of depth $(k - 1)$ has $n$ vertices, there is nothing more to prove. Otherwise it has $2^{k-1}s + 1$ vertices. Observe that for each set of $2^{k-1}s + 1$ vertices in $G_{n,p}$ the existence of a nice tree with this vertex set is independent from the existence of edges between vertices from this set and the remaining vertices. Let $T$ be a nice tree of depth $(k - 1)$.

We consider two cases. Case 1: $2^k s + 1 \leq n$. In this case there are at least $2^{k-1}s$ vertices in $G_{n,p}$ which are not vertices of $T$. Let $A$ be the union of the vertex sets of all branches of $T$ and $B$ be the complement of the vertex set of $T$. If we add to $T$ a matching $F_A$ satisfying the conditions of Lemma 1 (and its vertices), we get a nice tree of depth $k$, hence

$$\mathbb{P}_p(T_k | T_{k-1}) \geq \mathbb{P}_p(\text{existence of } F_A | T_{k-1}) = \mathbb{P}_p(\text{existence of } F_A) \geq 1 - \exp(-c_1 n^{1/2})$$

8
(we used the independence mentioned above). Hence
\[ P_p(T_k) \geq P_p(T_{k-1})(1 - \exp(-c_1n^\frac{1}{2})) \]
in this case.

**Case 2.** \(2^ks + 1 > n\). In this case \(k = d\), and the number of vertices in \(G_{n,p}\) which are not in \(T\) is \(< 2^{k-1}s\). We denote this set by \(A\) and let \(B\) the set of all vertices of \(T\) except \(v_0\). We apply Lemma 1 to these \(A\) and \(B\), and denote the corresponding matching by \(F_A\). Adding \(F_A\) (and its vertex set) to \(T\) we get a nice tree of depth \(d\). The probability is estimated in the same way as in Case 1. ■

**Problem 2** It would be interesting to find more precise estimates for the spanning tree congestion of random graphs. It seems plausible that “most” random graphs in \(G(n, p)\) satisfy \(s(G_{n,p}) \leq C \cdot n\) for some absolute constant \(C\).

**References**


