Unitarizable representations and fixed points of groups of biholomorphic transformations of operator balls
(joint work with V. S. Shulman and L. Turowska)

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The problem is: Under which conditions there is an invertible operator $V \in \mathcal{L}(\mathcal{H})$ such that the representation $\sigma$ of $G$, defined by the formula $\sigma(g) = V \pi(g) V^{-1}$, is unitary? (That is, all operators $V \pi(g) V^{-1}$, $g \in G$ are unitary.)
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The word *representation* means that $\pi(g^{-1}) = (\pi(g))^{-1}$ and $\pi(gh) = \pi(g)\pi(h)$ for all $g, h \in G$, where $\pi(g)\pi(h)$ is the composition of operators.
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Observations and known results

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In fact, if $\sigma(g) = V\pi(g)V^{-1}$ is unitary, then
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\pi(g) = V^{-1}\sigma(g)V \quad \text{and} \quad \|\pi(g)\| \leq \|V^{-1}\|\|\sigma(g)\|\|V\| = \|V^{-1}\|||V||.
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In fact, if $\sigma(g) = V\pi(g)V^{-1}$ is unitary, then $\pi(g) = V^{-1}\sigma(g)V$ and 
$$\|\pi(g)\| \leq \|V^{-1}\|\|\sigma(g)\|\|V\| = \|V^{-1}\|\|V\|.$$ 

Day and Dixmier proved that this condition is also sufficient for amenable groups (that is, groups admitting invariant means). Proof of this result is also based on averaging.
The simplest (in some sense) group known to have bounded non-unitarizable representations is the free group \( \mathbb{F}_2 \) with two generators. (The group \( \mathbb{F}_2 \) is a group of ‘words’ in an alphabet consisting of four symbols, \( a, b, a^{-1}, b^{-1} \), with multiplication defined as concatenation, with unit \( e \) defined as an empty word and with relations \( aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b = e \).)
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For this reason I do not discuss the mentioned problem, see N. Monod, N. Ozawa [The Dixmier problem, lamplighters and Burnside groups, J. Funct. Anal. (2009), doi:10.1016/j.jfa.2009.06.029] for a recent achievement in this direction and related references.
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We do not need any continuity assumptions on representations (for this reason I do not introduce them).
Our result is inspired by the theory of operators on spaces with an indefinite metric and algebras of operators on such spaces.
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We show that a bounded representation $\pi$ of a group $G$ on a Hilbert space $\mathcal{H}$ is similar to a unitary representation if it preserves a quadratic form $\eta$ with finite number of negative squares. The last condition means that $\eta(x) = \|Px\|^2 - \|Qx\|^2$ and $P, Q$ are orthogonal projections in $\mathcal{H}$ with $P + Q = 1$ (we use 1 to denote the identity operator) and $\dim(Q\mathcal{H}) < \infty$.

The result in the case when $\dim(Q\mathcal{H}) = 1$ was proved by Shulman (1980).
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We show that a bounded representation $\pi$ of a group $G$ on a Hilbert space $\mathcal{H}$ is similar to a unitary representation if it preserves a quadratic form $\eta$ with finite number of negative squares. The last condition means that $\eta(x) = \|Px\|^2 - \|Qx\|^2$ and $P$, $Q$ are orthogonal projections in $\mathcal{H}$ with $P + Q = 1$ (we use 1 to denote the identity operator) and $\dim(Q\mathcal{H}) < \infty$.

The result in the case when $\dim(Q\mathcal{H}) = 1$ was proved by Shulman (1980).
How do we use the quadratic from $\eta(x)$?

Let $K$ be the image of the orthogonal projection $Q$ and $H$ be the image of $P$. Then, as it was observed by M. G. Krein (1950), to each invertible operator preserving the quadratic form $\eta(x) = \|Px\|^2 - \|Qx\|^2$ there corresponds a biholomorphic mapping of the open unit ball $B = \{ T \in \mathcal{L}(K, H) : \|T\| < 1 \}$ of the space $\mathcal{L}(K, H)$. By a biholomorphic mapping we mean a bijective holomorphic mapping with holomorphic inverse.
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- Holomorphic mapping from an open subset in one Banach space into another Banach space can be defined in the same way as for $\mathbb{C}$: they are single-valued differentiable functions. In the infinite-dimensional case there are several natural notions of differentiability. It turns out that they lead to the same notion of a holomorphic function.
Krein’s observation

Recall some terminology of the theory of spaces with an indefinite metric. We introduce an *indefinite inner product* on $H$ by $[x, y] = (Px, y) - (Qx, y)$. A vector $x \in H$ is called *positive* (*neutral*, *negative*) if $[x, x] > 0$ ($[x, x] = 0$, $[x, x] < 0$, respectively). A subspace of $H$ is called *positive* (*neutral*, *negative*) if all its non-zero elements are positive (*neutral*, *negative*, respectively).
Recall some terminology of the theory of spaces with an indefinite metric. We introduce an indefinite inner product on $\mathcal{H}$ by $[x, y] = (Px, y) - (Qx, y)$. A vector $x \in \mathcal{H}$ is called positive (neutral, negative) if $[x, x] > 0$ ($[x, x] = 0$, $[x, x] < 0$, respectively). A subspace of $\mathcal{H}$ is called positive (neutral, negative) if all its non-zero elements are positive (neutral, negative, respectively).

For each operator $X \in \mathcal{B}$ the set

$$S(X) = \{Xx \oplus x : x \in K\}$$

is a negative subspace of $\mathcal{H}$ (recall that $\|X\| < 1$ for $X \in \mathcal{B}$). Since $\dim(S(X)) = \dim(K)$, $S(X)$ is a maximal negative subspace in $\mathcal{H}$. Indeed, if some subspace $M$ of $\mathcal{H}$ strictly contains $S(X)$, then its dimension is greater than the codimension of $H$, whence $M \cap H \neq \{0\}$. But all non-zero vectors in $H$ are positive.
Conversely, each maximal negative subspace $Q$ of $\mathcal{H}$ coincides with $S(X)$, for some $X \in \mathcal{B}$. Indeed, since $Q \cap H = \{0\}$, there is an operator $X : K \rightarrow H$ such that each vector of $Q$ is of the form $Xx \oplus x$. Since $Q$ is negative, we have

$$[Xx \oplus x, Xx \oplus x] = \|Xx\|^2 - \|x\|^2 < 0.$$ Since $K$ is finite dimensional, this implies $\|X\| < 1$, so $X \in \mathcal{B}$. Thus $Q \subset S(X)$; and, by maximality, $Q = S(X)$. 

It is easy to see that the map $X \rightarrow S(X)$ from $\mathcal{B}$ to the set $E$ of all maximal negative subspaces is injective and therefore bijective.

Recall that our purpose now is to establish a correspondence between invertible $\cdot, \cdot$-preserving operators and biholomorphic maps of $\mathcal{B}$. 

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- Recall that our purpose now is to establish a correspondence between invertible $[\cdot, \cdot]$-preserving operators and biholomorphic maps of $\mathcal{B}$.
Krein’s observation

Now we define the biholomorphic map \( w_U : \mathcal{B} \rightarrow \mathcal{B} \) corresponding to \([\cdot, \cdot]\)-preserving operator \( U \). Note that if a subspace \( L \) of \( \mathcal{H} \) is maximal negative, then its image \( U(L) \) under \( U \) is also maximal negative (because \( U \) is invertible and preserves \([\cdot, \cdot]\)). Hence, for each \( X \in \mathcal{B} \), there is \( Y \in \mathcal{B} \) such that \( S(Y) = US(X) \). We let \( w_U(X) = Y \).
Krein’s observation

Now we define the biholomorphic map $w_U : B \to B$ corresponding to $[\cdot, \cdot]$-preserving operator $U$. Note that if a subspace $L$ of $\mathcal{H}$ is maximal negative, then its image $U(L)$ under $U$ is also maximal negative (because $U$ is invertible and preserves $[\cdot, \cdot]$). Hence, for each $X \in B$, there is $Y \in B$ such that $S(Y) = US(X)$. We let $w_U(X) = Y$.

Now we write a formula for $U$ which can be used to show that $w_U$ is holomorphic. Let $U = (U_{ij})^2_{i,j=1}$ be the matrix of $U$ with respect to the decomposition $\mathcal{H} = H \oplus K$. Then $U(Xx \oplus x) = (U_{11}Xx + U_{12}x) \oplus (U_{21}Xx + U_{22}x)$. Since $U(Xx \oplus x) \in S(w_U(X))$, we conclude that

$$w_U(X)(U_{21}Xx + U_{22}x) = U_{11}Xx + U_{12}x.$$

Thus

$$w_U(X) = (U_{11}X + U_{12})(U_{21}X + U_{22})^{-1}. \quad (1)$$
It can be checked that the composition of the map $U \mapsto w_U$ and $\pi$ is a homomorphism of the group $G$ into the group of biholomorphic transformations on $B$. Now we show that if this group of biholomorphic transformations has a fixed point, then $\pi$ is unitarizable.
Relation with fixed points

- It can be checked that the composition of the map $U \mapsto w_U$ and $\pi$ is a homomorphism of the group $G$ into the group of biholomorphic transformations on $B$. Now we show that if this group of biholomorphic transformations has a fixed point, then $\pi$ is unitarizable.

- In fact, let $D \in B$ be such that $w_\pi(g)(D) = D$ for all $g \in G$. Hence $\pi(g)S(D) = S(D)$ for all $g \in G$.

  Let $U$ be an operator on $\mathcal{H}$ with the matrix $(U_{ij})$ where

  $U_{11} = (1_H - DD^*)^{-1/2}$, $U_{12} = -D(1_K - D^*D)^{-1/2}$,
  $U_{21} = -D^*(1_H - DD^*)^{-1/2}$, $U_{22} = (1_K - D^*D)^{-1/2}$.

  Straightforward computation shows that $U$ preserves $\eta$ and maps $S(D)$ onto $K$. Therefore all operators $\tau(g) = U\pi(g)U^{-1}$ preserve $\eta$, and the subspace $K$ is invariant for them. It follows that $H$ is also invariant for operators $\tau(g)$. Hence these operators preserve the scalar product on $\mathcal{H}$. Thus $g \mapsto \tau(g)$ is a unitary representation similar to $\pi$. 

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Looking for fixed points

The computation from the previous slide shows that to complete the proof we need to find a fixed point of the group \( \{w_\pi(g)\}_{g \in G} \) of biholomorphic mappings of \( B \). To achieve this goal we are going to use well-known tools of the theory of non-expansive mappings. Let us recall them.

Let \( (X, d) \) be a metric space. By a ball in \( (X, d) \) we mean a closed ball \( E(a, r) = \{x \in X : d(a, x) \leq r\} \). We say that \( (X, d) \) is ball-compact if a family of balls has non-void intersection provided each its finite subfamily has non-void intersection.

A subset \( M \subset X \) is called ball-convex if it is an intersection of a family of balls. It is clear from the definition that each ball-convex set is bounded and closed.

Lemma
Let \( (X, d) \) be ball-compact. A family \( \{M_\lambda: \lambda \in \Lambda\} \) of ball-convex subsets of \( X \) has non-empty intersection if each its finite subfamily has non-empty intersection.
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- A subset $M \subset X$ is called ball-convex if it is an intersection of a family of balls. It is clear from the definition that each ball-convex set is bounded and closed.

**Lemma**

*Let $(X, d)$ be ball-compact. A family $\{M_\lambda : \lambda \in \Lambda\}$ of ball-convex subsets of $X$ has non-empty intersection if each its finite subfamily has non-empty intersection.*
The **diameter** of a subset $M \subset X$ is defined by

$$\text{diam} M = \sup \{d(x, y) : x, y \in M\}.$$  \hfill (2)

A point $a$ in a bounded subset $M$ is called **diametral** if

$$\sup \{d(a, x) : x \in M\} = \text{diam} M.$$  

A metric space $X$ is said to have **normal structure** if every ball-convex subset of $X$ with more than one element has a non-diametral point.
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The concept of normal structure, introduced by Brodskii and Milman (1948) for Banach spaces, has played a prominent role in fixed point theory. For our purposes we need the following application of the normal structure.
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**Theorem**

*Suppose that a metric space $(\mathcal{X}, d)$ is ball-compact and has normal structure. If a group of isometries of $(\mathcal{X}, d)$ has a bounded orbit, then it has a common fixed point.*
Proof. Let $G$ be a group of isometries of $(\mathcal{X}, d)$ and let $G(x)$ be a bounded orbit, where $x$ is some point in $\mathcal{X}$. Then the family $\Phi$ of all balls containing $G(x)$ is non-empty. Since $G(x)$ is invariant under $G$, the family $\Phi$ is also invariant: $g(E) \in \Phi$, for each $E \in \Phi$. Hence the intersection $M_1$ of all elements of $\Phi$ is a non-void $G$-invariant ball-convex set.
Proof. Let $G$ be a group of isometries of $(\mathcal{X}, d)$ and let $G(x)$ be a bounded orbit, where $x$ is some point in $\mathcal{X}$. Then the family $\Phi$ of all balls containing $G(x)$ is non-empty. Since $G(x)$ is invariant under $G$, the family $\Phi$ is also invariant: $g(E) \in \Phi$, for each $E \in \Phi$. Hence the intersection $M_1$ of all elements of $\Phi$ is a non-void $G$-invariant ball-convex set.

Thus the family $\mathcal{M}$ of all non-void $G$-invariant ball-convex subsets of $\mathcal{X}$ is non-empty. It follows from the Lemma that the intersection of a decreasing chain of sets in $\mathcal{M}$ belongs to $\mathcal{M}$. By Zorn Lemma, $\mathcal{M}$ has minimal elements. Our aim is to prove that a minimal element $M$ of $\mathcal{M}$ consists of one point.
Assume the contrary and let \( \text{diam} M = \alpha > 0 \). Since \((\mathcal{X}, d)\) has normal structure, \( M \) contains a non-diametral point \( a \). It follows that \( M \subset \{ x \in \mathcal{X} : d(a, x) \leq \delta \} \) for some \( \delta < \alpha \). Set

\[
O = \bigcap_{b \in M} E_{b, \delta}.
\]

The set \( O \) is non-empty because \( a \in O \). Furthermore \( O \) is ball-convex by definition. To see that \( O \) is a proper subset of \( M \) take \( b, c \in M \) with \( d(b, c) > \delta \), then \( c \notin E_{b, \delta} \), hence \( c \notin O \).
Assume the contrary and let $\text{diam}M = \alpha > 0$. Since $(\mathcal{X}, d)$ has normal structure, $M$ contains a non-diametral point $a$. It follows that $M \subset \{x \in \mathcal{X} : d(a, x) \leq \delta\}$ for some $\delta < \alpha$. Set

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Since $G$ is a group of isometric transformations and $M$ is invariant under each element of $G$, the action of $G$ on $M$ is by isometric bijections. Therefore $O$ is $G$-invariant. We get a contradiction with the minimality of $M$. 

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Unitarization using fixed points
This result shows that to complete our argument it suffices to find a metric \( \rho \) on \( \mathcal{B} \) with respect to which biholomorphic mappings are isometries, the group \( \{ w_{\pi(g)} \} \) has a bounded orbit, and such that the metric space \((\mathcal{B}, \rho)\) is ball-compact and has normal structure.
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We prove that all these conditions hold when \( \rho \) is the Carathéodory metric on \( \mathcal{B} \).
This result shows that to complete our argument it suffices to find a metric $\rho$ on $\mathcal{B}$ with respect to which biholomorphic mappings are isometries, the group $\{w_{\pi(g)}\}$ has a bounded orbit, and such that the metric space $(\mathcal{B}, \rho)$ is ball-compact and has normal structure.

We prove that all these conditions hold when $\rho$ is the Carathéodory metric on $\mathcal{B}$.

The Carathéodory metric is an analogue of the Poincarè hyperbolic metric on the unit disk $\{z \in \mathbb{C} : |z| < 1\}$:

$$\pi(v, w) = \tanh^{-1}\left| \frac{v - w}{1 - \overline{w}v} \right|.$$
In the last statement I meant that for the Carathéodory metric on $\mathcal{B}$ we also can give a formula in terms of Möbius transformation:

$$\rho(A, B) = \tanh^{-1}(||M_{-A}(B)||),$$

where

$$M_A(X) = (1 - AA^*)^{-1/2}(A + X)(1 + A^*X)^{-1}(1 - A^*A)^{1/2}$$

is the Möbius transformation of the operator ball $\mathcal{B}$. 

It seems that Potapov (1950) was the first to suggest this formula for the Möbius transformation for operator balls.

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The fact that biholomorphic mappings of $\mathcal{B}$ are isometries for $\rho$ follows from the general theory of the Carathéodory metric.
Boundedness of orbits of the group \( \{ w_{\pi(g)} \}_{g \in G} \) can be derived from the assumption that \( \pi \) is a bounded representation by a direct computation.

Ball-compactness of \((B, \rho)\). Can be derived from reflexivity of \( L(K, H) \) (dim \( K \) < \( \infty \)) and the following observations:

1. Balls of \( \rho \) centered at 0 are the same of operator balls of the corresponding radii (follows from definitions).
2. The metric \( \rho \) is equivalent to the operator norm on any \( \rho \)-bounded set. (Well known and rather straightforward.)
3. Fractional-linear transformations (in particular, Möbius transformations) of the operator ball on Hilbert spaces map balls onto convex sets (it is interesting that this fact characterizes Hilbert spaces isometrically, Khatskevich-Shulman (1995)).
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To show that the space \((B, \rho)\) has normal structure it turns out to be convenient to introduce the following notion.

**Definition**

A metric space \((X, d)\) is said to have the **midpoint property** if for any two points \(a, b \in X\) there is \(c \in X\) such that
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d(c, x) \leq \frac{d(a, x) + d(b, x)}{2}
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\(\forall x \in (X, d)\).

Each point \(c\) satisfying this condition is called a **midpoint** for \((a, b)\), the set of all midpoints for \((a, b)\) is denoted by \(m(a, b)\).

A subset \(M\) of a metric space is called **mid-convex** if \(m(x, y) \subset M\) for each pair \((x, y)\) of points in \(M\).
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For mid-convex sets the negation of normal structure has a nice reformulation:

**Lemma**

Let $M$ be a bounded mid-convex subset of a metric space $(X, d)$ having the midpoint property. If all points of $M$ are diametral, then $M$ contains a net $\{c_\lambda : \lambda \in \Lambda\}$ with the property:

$$\lim_{\lambda} d(c_\lambda, x) = \text{diam} M$$

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The next step in the proof uses an analysis of properties of the Carathéodory metric in order to show that the metric space $(B, \rho)$ has the midpoint property.
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After that we use the Lemma to show that $(B, \rho)$ has the normal structure and apply the Theorem.